274 Curves on Surfaces, Lecture 26

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28 Even more about the geometry of skein relations

Recall that giving a hyperbolic structure to a surface Σ gives a (discrete, faithful) representation of $\pi_1(\Sigma)$ in $\mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$ up to conjugacy. If Σ has marked points, then we want cusps in the hyperbolic structure, which gives a collection of ideal points in $\partial\mathbb{H}^2$ (lifts of the cusps) acted on by $\pi_1(\Sigma)$. The number of such orbits should be finite and should satisfy some other conditions.

Decorating cusps gives a collection of horocycles in $\mathbb{H}^2$ acted on by $\pi_1(\Sigma)$. Recall that horocycles can be thought of as (positive) null vectors in the light-cone model or as elements of $\mathbb{R}^2/\{\pm 1\}$.

Now ignore twisting and suppose that we have an $\mathrm{SL}_2(\mathbb{R})$-representation of $\pi_1(\Sigma)$. These can be identified with certain $\mathbb{R}^2$-bundles over Σ with a flat connection (equivalently, an $\mathbb{R}^2$-local system). We want bundles whose transition functions lie in SL_2 (equivalently, SL_2-local systems).

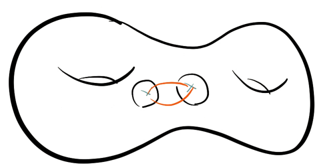

Figure 1: Monodromy of a bundle on a surface.

In this picture, a decorated cusp corresponds to a choice of vector in the fiber of the local system above each marked point.

$\mathrm{PSL}_2(\mathbb{R})$ acts on the unit tangent bundle $\mathrm{UT}(\mathbb{H}^2)$ freely and transitively, so $\mathrm{UT}(\mathbb{H}^2)$ can in fact be identified with $\mathrm{PSL}_2(\mathbb{R})$. Taking double covers gives $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{UT}^{(2)}(\mathbb{H}^2)$, and taking universal covers gives $\widetilde{\mathrm{SL}_2(\mathbb{R})} \cong \widetilde{\mathrm{UT}(\mathbb{H}^2)}$. (The universal cover of $\mathrm{SL}_2(\mathbb{R})$ is a good example of a group that has no faithful finite-dimensional representations.)

As we saw earlier, this story descends to Σ and gives both a canonical $\mathbb{Z}/2\mathbb{Z}$-extension and a canonical $\mathbb{Z}$-extension of $\pi_1(\Sigma)$, and this is how we define twisted representations. Also, as we saw earlier, a hyperbolic structure gives a PSL_2-representation which canonically lifts to a twisted SL_2-representation.

An immersed loop L on Σ gives a loop in $\mathrm{UT}(\Sigma)$. Given a twisted representation $\tilde{\rho}$, we can now extract a number $\mathrm{tr}(\tilde{\rho}(\tilde{L}))$.

Proposition 28.1. *If $\tilde{\rho}$ comes from a hyperbolic structure and L is taut, then $tr(\tilde{\rho}(\tilde{L})) > 2$.*

Proof. If L is taut, then it is regular isotopic to its geodesic representative L_2. Moreover, L_2 lifts to a curve $\tilde{L}_2$ in $\mathrm{UT}(\Sigma)$ which is still geodesic, and it lifts again to a geodesic (but not necessarily closed) curve in $\mathrm{UT}^{(2)}(\mathbb{H}^2) \cong \mathrm{SL}_2(\mathbb{R})$. A geodesic in a Lie group (with respect to a bi-invariant metric) is up to translation of the form $e^{tM}, M \in \mathfrak{sl}_2(\mathbb{R})$, and the number we want is $\mathrm{tr}(e^M)$. We know that e^M is hyperbolic, which means that the absolute value of the trace is greater than 2, and diagonalizing M the conclusion follows. □

Theorem 28.2. *The positive real points of $Spec(Sk(\Sigma))$ (the real points on which the bands or bracelets basis evaluate to positive numbers) are naturally identified with $Teich(\Sigma)$.*

Proof. (Sketch) The interesting case is when Σ is closed. Let $\nu : \mathrm{Sk}(\Sigma) \to \mathbb{R}$ be a positive point. If L is a simple loop, then $\nu(L) > 0$, but we also have $\nu(\mathrm{Brac}^{(k)})(L) > 0$. But

$$\nu(\mathrm{Brac}^{(k)}(L)) = T_k(\nu(L)) \tag{1}$$

where T_k is the k^{th} Chebyshev polynomial. The condition that this is positive for all k implies that $\nu(L) \geq 2$. The complex points of $\mathrm{Spec}(\mathrm{Sk}(\Sigma))$ can be identified with twisted $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(\Sigma)$ (Bullock), and ν itself gives a representation into $\mathrm{PSL}_2(\mathbb{R})$ in which all elements are parabolic or hyperbolic. This is not quite enough to show that ρ is discrete; there is more work needed...

Some indication of why this should be true. The closure of the image of ρ in $\mathrm{PSL}_2(\mathbb{R})$ is a Lie subgroup. Its connected component of the identity is a connected Lie subgroup, hence corresponds to some Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$. If the image of ρ consists of hyperbolic elements (and the identity) then the closure of the image cannot be all of $\mathrm{PSL}_2(\mathbb{R})$, so it suffices to rule out the other possible images. □

We now return to the case of marked points (on the boundary for simplicity). We would like to generalize twisted representations to this case. A twisted representation corresponds to an SL_2-local system, not on a surface Σ, but on its unit tangent bundle $\mathrm{UT}(\Sigma)$. We then associate to each marked point a vector in the fiber of the local system above the outward-pointing normal to the marked point.

We can now associate real numbers to arcs A between marked points p and q given a decorated local system as follows. By lifting the arc to $\mathrm{UT}(\Sigma)$ appropriately, we can

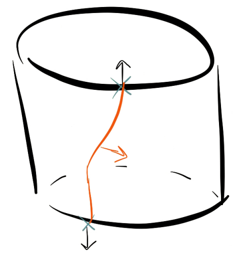

Figure 2: Vectors in the fibers above outward-pointing normals.

get a linear map $\tilde{\rho}(\tilde{A})$ from the fiber of the local system over $\tilde{p}$ (the outward-pointing normal at p) to $\tilde{q}$ (the outward-pointing normal at q). We now choose the real number

$$\tilde{\rho}(\tilde{A})v_p \wedge v_q \tag{2}$$

where v_p is the chosen vector over $\tilde{p}$ and v_q is the chosen vector over $\tilde{q}$. (We have chosen an identification of the determinant bundle with the trivial line bundle.)

Recall that the identity

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) \tag{3}$$

for $A, B \in \mathrm{SL}_2$ gives us the skein relations for loops.

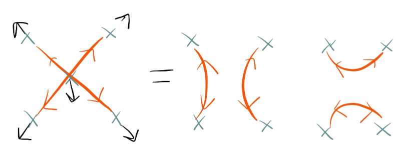

Figure 3: The skein relations again.

The skein relations for arcs can be obtained using the identity

$$Av \wedge w + A^{-1}v \wedge w = \mathrm{tr}(A)\,(v \wedge w) \tag{4}$$

and the Plücker relation

$$(v_1 \wedge v_3)(v_2 \wedge v_4) = (v_1 \wedge v_2)(v_3 \wedge v_4) + (v_1 \wedge v_4)(v_2 \wedge v_3). \tag{5}$$

For example, to prove the skein relation for two crossing arcs, we can translate everything to the fiber over the intersection point.