274 Curves on Surfaces, Lecture 25

Dylan Thurston
Notes by Qiaochu Yuan

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27 Types of multicurves

References for today: Mirzakhani, Growth of the number of simple closed geodesics on hyperbolic surfaces. Rivin, A simpler proof of Mirzakhani's simple curve asymptotics.

Thurston conjectured the following. Let M be a compact 3-manifold whose boundary is a union of tori. If M is irreducible, atoroidal, and has infinite π_1, then M has a finite cober which fibers over S^1. More generally, we might ask how common it is for a 3-manifold to fiber over S^1.

A 3-manifold has tunnel-number one if $M = H \cup (D^2 \times I)$ where H is an orientable handlebody of genus 2 and the two pieces have been glued along a simple closed curve γ on ∂H. We choose such a thing randomly by choosing Dehn-Thurston coordinates of the corresponding curve on ∂H randomly with size $\leq L$. As $L \to \infty$, it turns out that the probability that M fibers over the circle vanishes as $L \to \infty$.

Alternately, we could fix a set of generators of the mapping class group of ∂H (e.g. some Dehn twists) and randomly apply them to an initial curve γ_0. Conjecturally as $L \to \infty$ the probability that M fibers of the circle still vanishes as $L \to \infty$.

We want the curve γ above to be connected and non-separating. By this we mean the following. Consider multicurves in a surface of genus 2 up to the action of the mapping class group (types of multicurves). A connected such curve either divides the surface into two genus 1 pieces (the separating case) or loops around one of the two holes (the non-separating case), and the general multicurve is a union of such things.

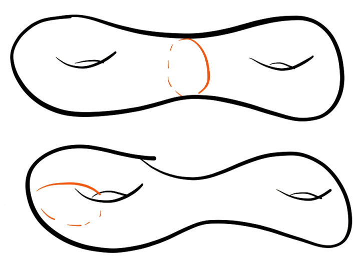

Figure 1: Connected curves on a two-holed torus.

Theorem 27.1. *(Mirzakhani) Fix a multicurve γ. The probability that a random multicurve in Dehn-Thurston coordinates is equivalent to γ under the action of the mapping class group approaches a limit $0 < c_\gamma < 1$ as $L \to \infty$. Furthermore, if $\Omega \subset \mathbb{R}^{6g-6}$ is a bounded region in the space of Dehn coordinates, the proportion of Dehn-Thurston coordinates of random curves that sit inside Ω after rescaling and* *that are equivalent to γ under the action of the mapping class group again, suitably rescaled, again approaches c_γ.*

In the case that γ is connected and nonseparating we have $c_\gamma \approx \frac{1}{5}$.

Compare to the case $g = 1$. Then there is only one type of connected curve, and a simple multicurve up to the action of the mapping class group is a finite number of copies of this.

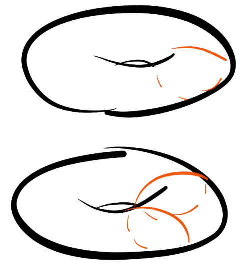

Figure 2: Curves on a torus.

Choosing a random multicurve on the torus means choosing a random pair (p, q) of positive integers, and the number of connected components of the resulting curve is $\gcd(p, q)$. Mirzakhani's result in this case (which is much older) says that there is a definite probability of obtaining $\gcd(p, q) = 1$, which is just $\frac{6}{\pi^2}$.

The appearance of π here is not surprising. Another part of Mirzakhani's result is that c_γ is proportional to the Weil-Petersson volume of the Teichmüller space of $S \setminus \gamma$. (We consider punctures at the boundary components of $S \setminus \gamma$.) The Teichmüller space of $S_{g,n}$ has a canonical symplectic form ω, and ω^{3g-3+n} gives a canonical volume form.

(Edit: there is a result of this form, but the result above is not true as stated.)

To obtain ω, there is another set of coordinates on Teichmüller space called Feichel-Nielsen coordinates obtained by choosing a pants decomposition and looking at lengths ℓ_i of each curve, then looking at the twists t_i around each curve. The symplectic form is then $\omega_{WP} = \sum d\ell_i \wedge dt_i$ (in particular the above does not depend on the choice of pants decomposition).

Alternately, if the surface has a triangulation, then consider shear coordinates s_i for each edge of a triangulation T. Then

$$\omega_{WP} = \sum_{\Delta_{ijk}} (ds_i \wedge ds_j + ds_j \wedge ds_k + ds_k \wedge ds_i) \tag{1}$$

where the sum runs over all triangles and i, j, k are the edges in clockwise order (in particular the above does not depend on the choice of triangulation).

Theorem 27.2. *(Mirzakhani) The Weil-Petersson volume of the Teichmuller space of $S_{g,n}$ is a rational multiple of $\pi^{6g-6+2n}$.*

A key ingredient is that the action of the mapping class group on measured laminations is ergodic with respect to Lebesgue measure. (We say that the action of a group G on a measure space (X, μ) is ergodic if any G-invariant set is either empty or has full measure, and moreover any G-invariant measure that is absolutely continuous with respect to μ is a constant multiple of μ.)

When we compactified Teichmüller space, we tropicalized λ-lengths and obtained bounded measured laminations. Alternately, we tropicalized shear coordinates and obtained unbounded measured laminations. The latter does not give a symplectic manifold, but we can consider the subspace where the sum of the shear coordinates around each puncture is 0 (no spiraling into punctures).

Mirzakhani's result above can be translated into the theory of cluster algebras as follows.

Theorem 27.3. *(Mirzakhani) Fix a surface cluster algebra, not of finite type, with some set of marked points $m_1, ...m_k$. Consider random basis elements x with $deg_{m_i}(x) = 0$. Then the probability that x is of some type (e.g. connected) is definite (strictly between 0 and 1).*

A similar statement should be true for other mutation-finite cluster algebras (neither finite type nor affine). Mutation sequences giving a cluster with the same quiver form a group analogous to the mapping class group, and we can study the orbits of some conjectural positive basis under this group. Conjecturally the orbits are finitely generated in a suitable sense, there is a definite probability of getting any orbit, and the ratios of these probabilities are rational.

What happens in the non-mutation-finite case? What is the analogue of cutting a surface along a simple curve?