274 Curves on Surfaces, Lecture 24

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26 Monoidal categorification of skein algebras

Let C be an abelian category (an Ab-enriched category with certain nice properties). Its Grothendieck group $K_0(C)$ is the quotient of the free abelian group on isomorphism classes of objects in C by a relation $[B] = [A] + [C]$ for every short exact sequence

$$0 \to A \to B \to C \to 0 \tag{1}$$

in C. If moreover C is monoidal (with monoidal structure suitably compatible with the abelian structure) then $K_0(C)$ inherits a multiplication $[A][B] = [A \otimes B]$ and hence becomes a ring.

If furthermore C has simple objects and each object of C has a composition series of finite length, then $K_0(C)$ is freely generated by the simple objects, and if $X = X_i \supset X_{i-1} \supset ... \supset X_0 = 0$ is a composition series then

$$[X] = \sum_i [X_i / X_{i-1}]. \tag{2}$$

For example, $C = R$-Mod for R a commutative ring satisfies the above. We can also consider representations of groups. More generally we can take $C = H$-Mod where H is a Hopf algebra.

Under the above assumptions, $K_0(C)$ has a strongly positive basis given by simple objects. The structure constants are given by finding the composition series of the tensor product of two simple objects.

Example Consider the representation theory of $\mathrm{SL}_2(\mathbb{R})$. There is an obvious simple module $V = \mathbb{R}^2$. Its symmetric powers $V_k = \mathrm{Sym}^k(V)$ are also simple and can be identified with homogeneous polynomials of degree k in two variables. The tensor products are

$$V_k \otimes V_\ell \cong V_{k+\ell} \oplus V_{k+\ell-2} \oplus ... \oplus V_{|k-\ell|}. \tag{3}$$

Whenever we find an algebra with a positive integral basis, we can conjecture that that algebra has a monoidal categorification in the sense that we can identify it with $K_0(C)$ for some C such that the simple objects are sent to the integral basis. In particular we conjecture that skein algebras $\mathrm{Sk}(\Sigma)$ have this property with the simple objects sent to the bracelets basis.

In the monoidal categorification, product in $\mathrm{Sk}(\Sigma)$ corresponds to tensor product and unions of simple arcs correspond to simple objects. By contrast, in the additive categorification, product in $\mathrm{Sk}(\Sigma)$ corresponds to direct sum and simple arcs correspond to objects satisfying $\mathrm{Ext}^1(X, X) = 0$.

This conjecture is known for the surfaces whose quivers have type A_n and D_n but open in general. One reason to expect it to be true comes from the study of topological quantum field theories. A topological quantum field theory is a symmetric monoidal functor from a cobordism category to a category of vector spaces. Explicitly, for fixed n, it should assign vector spaces $Z(M)$ to $(n-1)$-manifolds M and, to an n-manifold with boundary $\bar{M} \sqcup N$, it should assign a linear map $Z(M) \to Z(N)$ in a way which is compatible with composition and tensor product.

For example, when $n = 2$ a topological quantum field theory is precisely a vector space $Z(S^1)$ together with a commutative product (coming from the pair of pants) and a linear functional (coming from half of a torus) such that

$$\langle ab, c\rangle = \langle a, bc\rangle \tag{4}$$

and which is nondegenerate, and this is precisely a (commutative) Frobenius algebra.

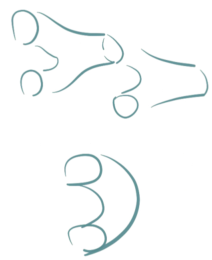


Figure 1: Structure maps of a Frobenius algebra.

When $n = 3$ there are a rich class of examples coming from quantum groups. Associated to any semisimple Lie group G is a quantum group $U_q(\mathfrak{g})$ depending on a prameter q, and setting q to a root of unity we can obtain Witten-Reshetikhin-Turaev TQFTs.

Rather than construct functors out of cobordism categories we can construct functors out of the category of framed tangles.

One such functor is essentially the Jones polynomial. That is, it is obtained by taking free linear combinations and quotienting by the skein relations, which gives an intermediate category called the Temperley-Lieb category, and then applying $\text{Hom}(\emptyset, -)$. The Temperley-Lieb category is spanned by tangles with no crossings. This can be used to construct an $n = 3$ TQFT.

When $n = 4$ we have an interesting near-TQFT (modulo conjectures): Donaldson theory is dual to Seiberg-Witten theory is equivalent to embedded contact homology

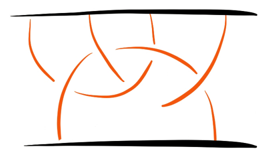


Figure 2: A tangle.

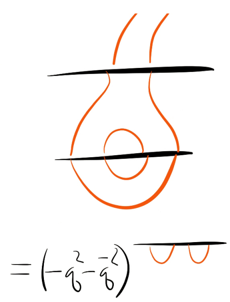


Figure 3: Composition in the Temperley-Lieb category.

is equivalent to Hegaard-Floer homology (these are all very difficult results). This is the only known way to distinguish smooth structures on topological 4-manifolds, and it would be nice if it were easier to compute.

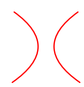
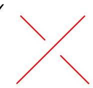
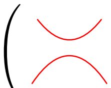
One way to compute is to extend downward. Currently these TQFTs associate data to 4-manifolds and 3-manifolds, but if they also associated data to 2-manifolds we could cut up the 3-manifolds further. In fact they associate homologies to 3-manifolds and can be made to associate derived categories to 2-manifolds.

More functorially, there is a 2-category whose objects are $n-2$-manifolds, whose morphisms are $n-1$-dimensional cobordisms between them, and whose 2-morphisms are n-dimensional cobordisms between morphisms (hence in particular are manifolds with corners), and we want a (symmetric monoidal) 2-functor out of this category to some linear 2-category, e.g. a 2-category of linear categories. For example, bordered Heegaard-Floer theory assigns to a surface Σ the derived category of modules over some dg-algebra $A(\Sigma)$, assigns to a 3-manifold a dg-bimodule, and which assigns to a 4-manifold a morphism of bimodules.

Conjecturally cluster algebras can be used to construct a different 4-dimensional

TQFT, and monoidal categorifications might be relevant to doing this. Even more conjecturally we should be able to monoidally categorify $\mathrm{Sk}_q(\Sigma)$ (although a strongly positive basis is not known in general in this case).

This is related to categorifying the Jones polynomial to Khovanov homology. Khovanov homology can be thought of as a functor from the category of framed knots in S^3 and cobordisms to bigraded vector spaces which recovers the Jones polynomial after taking Euler characteristics (alternating sum of graded dimensions). It upgrades the skein relation to a short exact sequence

$0 \to \mathrm{CKh}$ () $\to \mathrm{CKh}$ () $\to \mathrm{CKh}$ () $\to 0$ (5)

(with grading shifts) by constructing a chain complex using all possible resolutions of the crossings of a knot.

The 3-manifold TQFTs discussed above come from quantum groups. It is also possible to use quantum groups, together with a representation, to construct a knot polynomial, and Khovanov homology suggests that this story can be categorified. For example, quantum SL_2 and its standard representation gives the Jones polynomial, which is categorified to Khovanov homology. More generally, quantum SL_n and its standard representation gives the HOMFLYPT polynomial, which is categorified to Khovanov-Rozansky or HOMFLYPT homology. Similarly, quantum $\mathrm{SO}(n)$ and its standard representation gives the Kauffman polynomial (not to be confused with the Kauffman bracket), which is categorified to $\mathrm{SO}(n)$ homology (conjecturally).

We also get an invariant of tangles, at least for Khovanov homology, which should give a monoidal categorification for the skein algebra of the disk with n punctures.