274 Curves on Surfaces, Lecture 23

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25 More about additive categorification of surface cluster algebras (Christof)

Mutation of quivers with potential proceeds as follows. Consider again the quiver Q given by

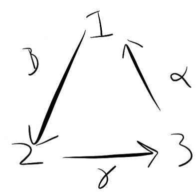

Figure 1: A quiver.

with potential $W = \gamma\beta\alpha$. We want to mutate at 1. The premutation $\tilde{Q}$ has new composite arrows for each composition of arrows through 1, and in addition we change the orientation of arrows starting or ending at 1, giving

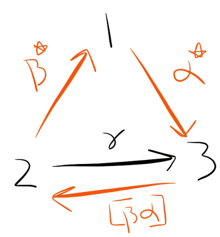

Figure 2: The premutation.

We now need to modify the potential by replacing $\beta\alpha$ with the composite arrow and adding a new term for each composite arrow, giving

$$\tilde{W} = \gamma[\beta\alpha] + [\beta\alpha]\alpha^*\beta^*. \tag{1}$$

The potential now has a 2-cycle, which we will try to eliminate as follows. There is an automorphism

$$\varphi : \gamma \mapsto \gamma + \beta^* \alpha^* \tag{2}$$

of the completed path algebra with the property that if $\tilde{W}' = \gamma[\beta\alpha]$, then $\varphi(\tilde{W}') = \tilde{W}$ up to cyclic rotation of summands. This allows us to remove the 2-cycle $\gamma[\beta\alpha]$, which trivializes the potential.

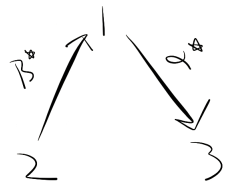

Figure 3: The mutation.

We can associate a quiver with potential to a triangulation of a surface as follows. We restrict our attention to ideal triangulations without tags or self-folded triangles. Details can be found in Labardini-Fragoso.

Let (Σ, M) be a surface with marked points M, a subset $P \subseteq M$ of which are punctures. Fix $x_p \in \mathbb{C}^*$ for each $p \in P$. We associate to a triangulation τ a non-reduced quiver $\tilde{Q}(\tau)$ with vertices given by the edges of the triangulation and edges associated to each interior triangle and each puncture.

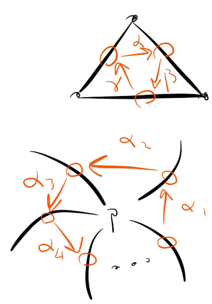

Figure 4: A triangle and a puncture.

We associate to each interior triangle a term $c_\Delta = \gamma\beta\alpha$ and to each puncture a term $c_p = \alpha_n ... \alpha_2\alpha_1$, and we sum these up to obtain a potential

$$\tilde{W}(\tau) = \sum_{\Delta} c_{\Delta} + \sum_{p} x_p c_p. \tag{3}$$

Finally, we need to remove 2-cycles. The resulting definition of a quiver with potential is compatible with flips of triangulations except in a few cases (Labardini-Fragoso), namely

1. $g(\Sigma) \geq 1$, $\partial\Sigma = \emptyset$ and $|M| \in \{2, 3, 4, 5\}$ and
2. $\Sigma = S^2$ and $|M| \in \{5, 6, 7, 8\}$.

Example Let Σ be a once-punctured 4-gon. The corresponding quiver has a 2-cycle. Before it is removed the potential is $\tilde{W} = \beta\alpha\varepsilon_2 + \delta\gamma\varepsilon_1 + x\varepsilon_2\varepsilon_1$, and after it is removed the potential is $W = \delta\gamma\beta\alpha$.

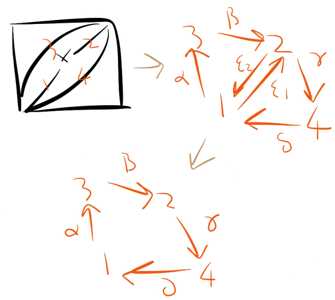

Figure 5: A quiver mutation on a surface.

To categorify surface cluster algebras we now need to associate modules to curves on the surface. We restrict our attention to the case that there are no punctures.

Example Consider a torus with a hole and a marked point on the boundary. The potential is

$$W = \beta^+\beta\beta^- + \alpha^+\alpha\alpha^-. \tag{4}$$

We want to associate modules to curves on this surface. Take, for example, a loop around the hole. We will follow the loop around and associate to each intersection we find a basis vector of the module. Each intersection is connected to the next intersection by some edge in the quiver, so the action of the path algebra is specified by sending intersections to the next intersection in this way.

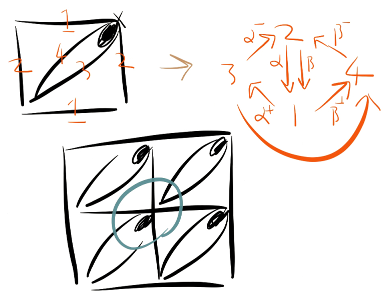

Figure 6: A curve on a surface.