274 Curves on Surfaces, Lecture 2

Dylan Thurston
Notes by Qiaochu Yuan

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2 Mapping class groups

Regarding the exercises from the previous lecture: it is cleaner to compactify and identify the point at infinity with the top and bottom punctures, so we work instead with a four-punctured sphere. Then there can be components of a curve which loop around one puncture or components which loop around two punctures (circumferences). This gives five different kinds of curves, and up to the action of the mapping class group every simple (multi)curve consists of copies of these five kinds of curves.

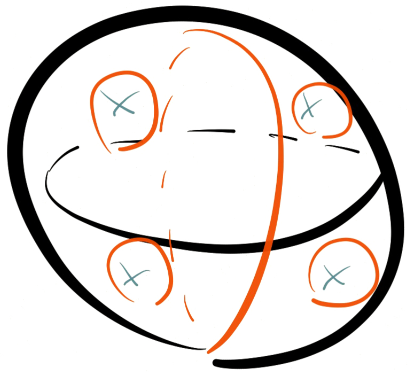
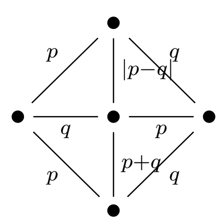


Figure 1: The five kinds of components of a simple multicurve on the four-punctured sphere.

In the first exercise, the three intersection numbers on the edges of a triangle must have the form $a+b, b+c, c+a$ where a, b, c are non-negative integers. This is equivalent to the three numbers summing to an even integer and satisfying the triangle inequality. Now, if all of the triangle inequalities among the coordinates are strict, then the curve necessarily has a component which loops around one puncture. Removing all such loops, we can work out that the coordinates must have the form



(1)

and the mapping class group acts essentially by Euclid's algorithm on p, q. This is not a coincidence; the mapping class group B_3 is related to $\mathrm{SL}_2(\mathbb{Z})$, which is in turn related to the Euclidean algorithm.

An easier case to handle first is the (orientation-preserving) mapping class group $\mathrm{MCG}^+(T^2) = \mathrm{Diff}^+(X)/\mathrm{Diff}_0(X)$ of the torus (where $\mathrm{Diff}_0(X)$ is the connected component of the group of diffeomorphisms containing the identity). This is the same as the mapping class group of the torus minus a point x, which acts on loops based at x.

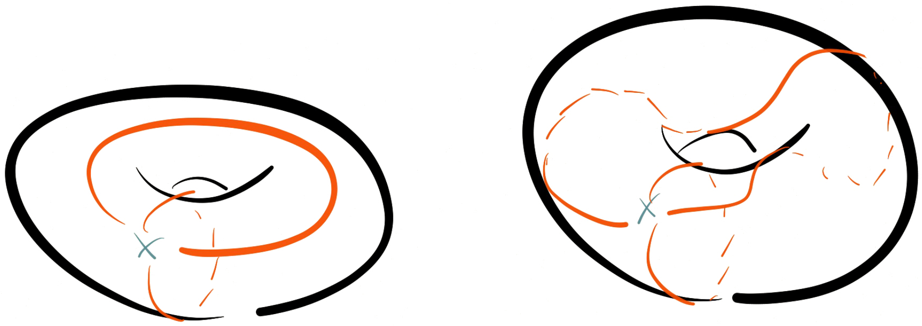


Figure 2: Two curves on a torus being acted on by the mapping class group.

This gives an action on the fundamental group, hence a homomorphism

$$\mathrm{MCG}(T^2, x) \to \mathrm{Aut}(\pi_1(T^2, x)) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}_2(\mathbb{Z}) \tag{2}$$

whose restriction to $\mathrm{MCG}^+(T^2, x)$ lands in $\mathrm{SL}_2(\mathbb{Z})$. This relies on the fact that $\pi_1(T^2, x)$ is abelian. In general we only get an action by outer automorphisms

$$\mathrm{MCG}(X) \to \mathrm{Out}(\pi_1(X)) \tag{3}$$

because the mapping class group acts on unbased loops, and changing the basepoint changes the corresponding automorphism of fundamental groups (based at different points) by conjugation by a path connecting the basepoints.

In the special case of surfaces Σ, the map $\mathrm{MCG}(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma))$ is an isomorphism. This is false in general. It is also false that the mapping class group of a space is the mapping class group of a space minus a point x. There is only a commutative diagram

$$\begin{tikzcd} \mathrm{MCG}(X) \arrow[r] & \mathrm{Out}(\pi_1(X)) \\ \mathrm{MCG}(X, x) \arrow[r] \arrow[u] & \mathrm{Aut}(\pi_1(X)) \arrow[u] \end{tikzcd} \tag{4}$$

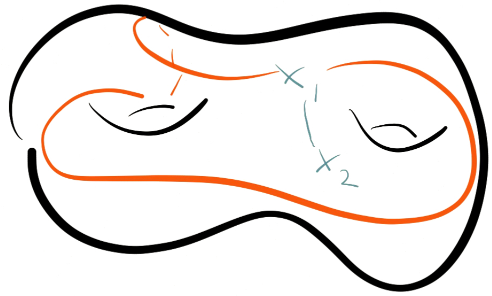


Figure 3: Changing basepoints on a surface with nonabelian fundamental group.

and the various maps in it are not isomorphisms in general, for example when X is a wedge of two circles $S^1 \vee S^1$. In this case MCG(X) has 8 elements (we can flip either of the two circles or switch the circles), MCG(X, x_0) = MCG(X) if x_0 is the wedge point, and MCG(X, x_1) has two elements if x_1 lies on only one of the circles. Here $\pi_1 = \mathbb{Z} * \mathbb{Z}$ is not abelian, so the map $\mathrm{Aut}(\pi_1) \to \mathrm{Out}(\pi_1)$ is also not an isomorphism.

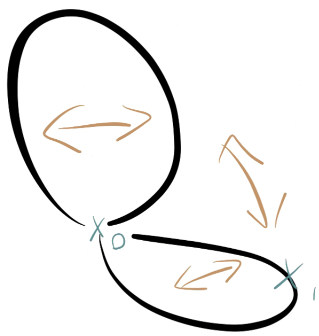


Figure 4: The wedge of two circles and the generators of its mapping class group.

Theorem 2.1. *$MCG(T^2) \cong MCG(T^2, x) \cong GL_2(\mathbb{Z})$ and $MCG^+(T^2) \cong MCG^+(T^2, x) \cong SL_2(\mathbb{Z})$.*

Regarding the distinction between $\mathrm{Aut}(\pi_1(X))$ and $\mathrm{Out}(\pi_1(X))$, we have the following result.

Theorem 2.2. *If G is a topological group, $\pi_1(G, e)$ is abelian.*

In particular T^2 is a topological group.

Exercise 2.3. *Find out what the Birman exact sequence is and report back next week.*

Back to curves. How can we describe curves on a punctured torus? We can triangulate and then count intersections as before. There are only three coordinates a, b, c, and there is a component around the puncture if various triangle inequalities are strict.

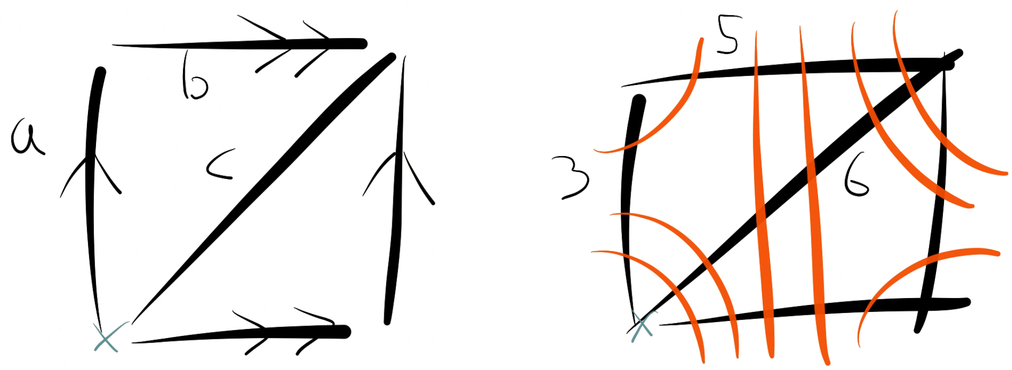


Figure 5: General coordinates for a curve on a punctured torus, and a specific curve with components around the puncture.

Removing all such components, assume WLOG that c is the greatest coordinate; then $c = a + b$. In this case curves are parameterized by two numbers (a, b), which in the universal cover $\mathbb{R}^2$ of the torus we can think of as given by a line with slope $\frac{b}{a}$ (when a, b are relatively prime).

The number of components of the corresponding curve is $\gcd(a, b)$ (the mapping class group acts by the Euclidean algorithm); consequently, we get a curve with a single component if and only if $\gcd(a, b) = 1$, so we can identify the simple closed curves on T^2 (not trivial, and not around the puncture) with the projective line $\mathbb{P}^1(\mathbb{Q})$ over $\mathbb{Q}$. The probability that $\gcd(a, b) = 1$ occurs asymptotically is

$$\frac{6}{\pi^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{-1}. \tag{5}$$

(Heuristically this is because the probability that $\gcd(a, b) = n$ should be proportional to $\frac{1}{n^2}$.)

Exercise 2.4. *Explain how running Euclid's algorithm corresponds to changing the triangulation.*

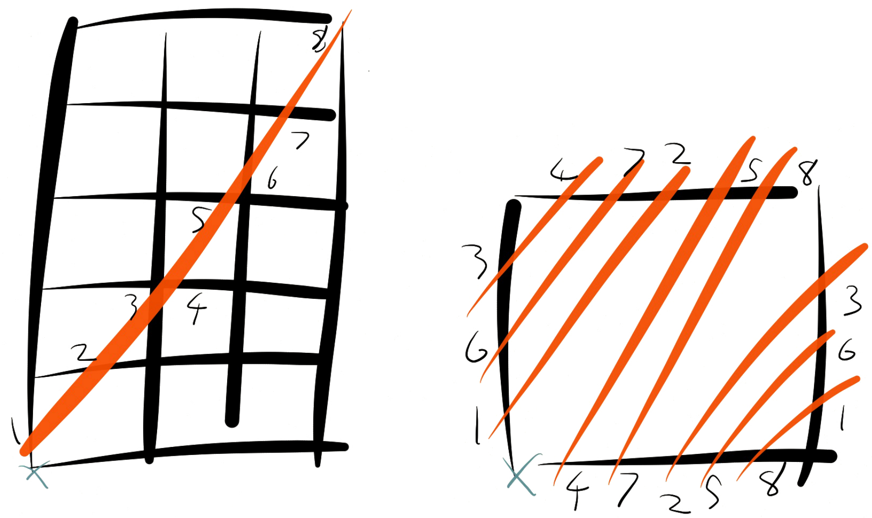


Figure 6: A line of slope $\frac{5}{3}$ in $\mathbb{R}^2$ and the corresponding curve on the torus.

Back to the braid group. What does $\mathrm{MCG}^+(T^2)$ have to do with the thrice-punctured disc or the four-punctured sphere? If Y denotes the four-punctured sphere, the double cover branched at the punctures $\tilde{Y}$ is a torus. Branched means that near the punctures the map looks locally like $z^2 \mapsto z$ in $\mathbb{C}$.

Figure 7: A local picture of the map $z^2 \mapsto z$ (given by projection down).

To obtain this double cover, skewer the torus by a line and quotient by rotation by 180° about the line. The corresponding cover is branched at the four points where we skewered the torus.

More algebraically, we quotiented by the element $-I$ in the mapping class group

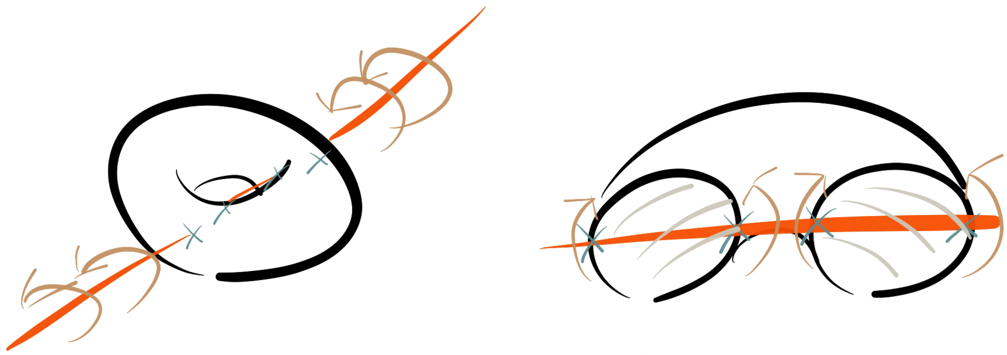


Figure 8: A skewered torus and a picture of the quotient, except that we still need to identify parts of the circles.

$\mathrm{SL}_2(\mathbb{Z})$. This branched cover descends to a map of mapping class groups

$$\mathrm{MCG}^+(T^2) \cong \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{MCG}^+(S^2, 1\text{ pt}, 3\text{ pts}). \tag{6}$$

The last mapping class group fixes one puncture pointwise and fixes the other three setwise. One way to see this action on three points is to look at the quotient $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_2)$.

On the other hand, there is a map

$$B_3 \cong \mathrm{MCG}(D^2, 3\text{ pts}) \to \mathrm{MCG}^+(S^2, 1\text{ pt}, 3\text{ pts}) \tag{7}$$

(where the first mapping class group fixes the boundary pointwise) and the claim is that this exhibits B_3 as a central extension of $\mathrm{PSL}_2(\mathbb{Z})$.