274 Curves on Surfaces, Lecture 17

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18 The Laurent phenomenon (Kalman)

The Laurent phenomenon is a phenomenon by which various recurrences defined by rational functions turn out to be Laurent polynomials in the first few terms. If the first few terms are set to 1, then the remaining terms become integers even though the recurrence divides by previous members of the term. For example, consider the sequence

$$y_k = \frac{y_{k-3}y_{k-1} + y_{k-2}^2}{y_{k-4}} \tag{1}$$

with initial conditions $y_1 = y_2 = y_3 = y_4$ (Somos-4).

Cluster algebras provide a natural setting for studying the Laurent phenomenon via the exchange relation (Fomin, Zelevinsky).

Theorem 18.1. *In a cluste ralgebra, any cluster variable is expressed in terms of any given cluster as a Laurent polynomial with coefficients in the group ring* $\mathbb{ZP}$.

We will instead prove a more general result, the Caterpillar Lemma. This is a statement about a sequence $\mathbb{T}_{n,m}$ of trees. This has m vertices of degree n in its spine.

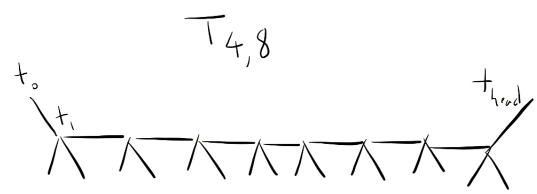

Figure 1: The caterpillar $T_{4,8}$.

We will label the edges emanating from each vertex with different labels and we will associate an exchange polynomial $P \in \mathbb{A}[x_1, ...x_n]$, not involving x_k, to every edge (here $\mathbb{A}$ is a UFD). This is a generalized exchange pattern.

Associate to t_0 the initial cluster $\mathbf{x}(t_0)$ of n independent variables. To each vertex $t \in \mathbb{T}_{n,m}$, we associate a cluster $\mathbf{x}(t)$. The variables in this cluster are uniquely determined by the exchange relations, for an edge labeled by k and P:

$$x_i(t) = x_i(t'), i \neq k \tag{2}$$

$$x_k(t)x_k(t') = P(\mathbf{x}(t)). \tag{3}$$

Lemma 18.2. *(Caterpillar) Suppose a generalized exchange pattern on $\mathbb{T}_{n,m}$ satisfies the following conditions:*

1. *for any edge labeled by k and P, the polynomial P does not involve x_k and is not divisible by any x_i.*
2. *For any two consecutive edges labeled by i, P and j, Q, the polynomials P and $Q_0 = Q|_{x_i=0}$ are coprime.*
3. *For any three consecutive edges labeled by i, P and j, Q and i, R, we have*

$$LQ_0^b P = R_{x_j \leftarrow \frac{Q_0}{x_j}} \tag{4}$$

where b is a non-negative integer and L is a Laurent monomial whose coefficients lie in $\mathbb{A}$ and which is coprime to P.

Then each $x_i(t), t \in \mathcal{T}_{n,m}$ is a Laurent polynomial in $x_1(t_0), \ldots x_n(t_0)$ with coefficients in $\mathbb{A}$.

Proof. For $t \in \mathcal{T}_{n,m}$ let $L(t)$ be the ring of Laurent polynomials in $\mathbf{x}(t)$. Abbreviate $L_0 = L(t_0)$. We proceed by induction on m (the length of the spine). This is straightforward for $m = 1$, so assume $m \geq 2$ and that the statement is true for all caterpillars with smaller spine.

We will need a lemma. Suppose the path from t_0 to t_{head} starts with edges labeled by i and j and consider the unique next edge labeled by i. Then the clusters $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3)$ are all in L_0, and in addition $\gcd(x_i(t_3), x_i(t_1)) = \gcd(x_j(t_2), x_i(t_1)) = 1$. The proof of this last statement involves the third condition.

Returning to the proof, by the inductive hypothesis $X = x_k(t_{\text{head}})$ belongs to both $L(t_1)$ and $L(t_3)$. We further claim that $X = \frac{f}{x_i(t_1)^a}$ for some $f \in L_0$ and some $a \geq 0$. This follows from the fact that $X \in L(t_1)$ and that $x_i(t_1) = \frac{P}{x_i} \in L_0$.

Similarly, we claim that $X = \frac{g}{x_j(t_2)^b x_i(t_3)^c}$ for some $g \in L_0$ and some $b, c \geq 0$. This follows from the fact that $X \in L(t_3)$, the fact that $x_i(t_3), x_j(t_3) \in L_0$ by the lemma, and the fact that $x_j(t_3) = x_j(t_2) \in L_0$.

We conclude that

$$X = \frac{f}{x_i(t)^a} = \frac{g}{x_j(t_2)^a x_i(t_3)^c}. \tag{5}$$

From the second part of the lemma, $\gcd(x_i(t_3), x_i(t_1)) = 1, \gcd(x_j(t_2), x_i(t_1)) = 1$, so $X \in L_0$ as desired. □

Good news: the Caterpillar Lemma can prove Laurentness in many situations. Bad news: it is often not trivial to rephrase a given problem in the Caterpillar Lemma framework. For example, when describing the caterpillar graph for the Somos-4 sequence, it is not obvious what the exchange polynomials on the legs should be. See Fomin-Zelevinsky for details.

19 Miscellaneous

Some sequences exhibiting the Laurent phenomenon like the Somos-7 sequence

$$x_k x_{k+7} = x_{k+1}x_{k+6} + x_{k+2}x_{k+5} + x_{k+3}x_{k+4} \tag{6}$$

cannot be described using cluster algebras (we would need the RHS to consist of two terms). Lam and Pylyavskyy have a notion of Laurent phenomenon algebra that addresses this.

Consider surfaces with boundary components and no marked points.

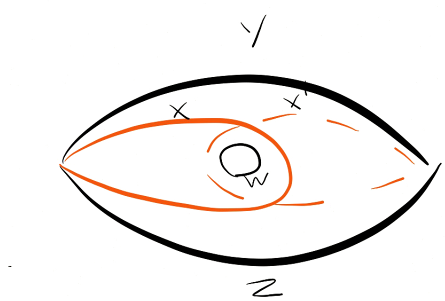

Figure 2: A surface with boundary components and no marked points.

We can write down what the exchange relations for the corresponding cluster algebra looks like using the skein relation.

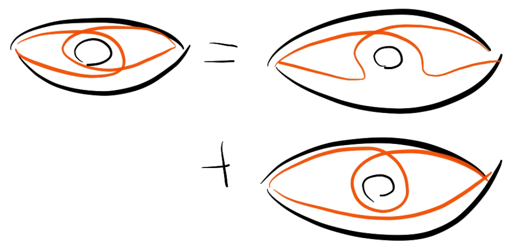

Figure 3: An application of the skein relation.

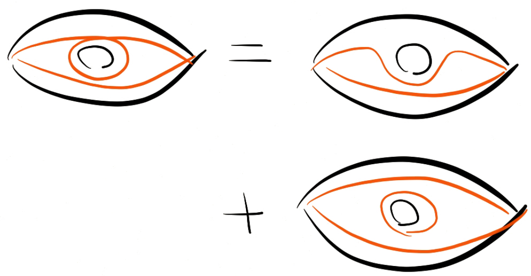

Figure 4: Another application of the skein relation.

This gives an exchange relation with three terms on the RHS (so not a cluster algebra): more specifically,

$$xx' = y^2 + z^2 + yzw. \tag{7}$$

There are two special cases. When $w = 0$, this has two terms. When $w = 2$, we should take $\sqrt{x}, \sqrt{x'}$ as the cluster variables, and then their product is $y + z$.

More geometrically, recall that for closed curves the λ-length is $2\cosh\left(\frac{\ell}{2}\right)$, or the trace of the monodromy. When $w = 2$ we have $\ell = 0$ (a cusp) and when $w = 0$ we are instead at a cone point and we should replace the hyperbolic cosine with the usual cosine of the angle, which is π at an order 2 cone point. In the case that $w = 2$, we added square roots because we are looking at a self-conjugate horocycle.

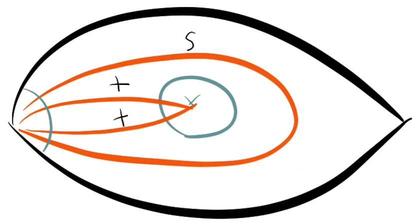

Figure 5: Self-conjugate horocycles and square roots. Here $t = \sqrt{s}$.

Let's return to skein theory. Last time we forgot an extra relation, namely that a loop around a puncture evaluates to 2.

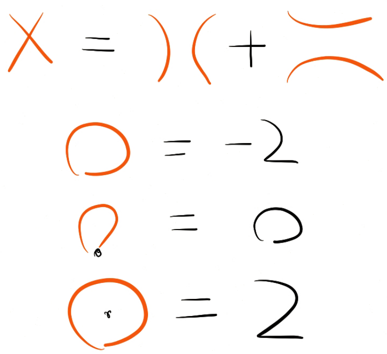

Figure 6: Relations for skein theory.

Not including this relation gives a sensible skein algebra, but one which has a zero divisor.

With this extra relation it is not completely obvious that these relations are consistent (that is, that the basis is still what we expect it to be). This can be proven using the diamond lemma (Bergman). This is a lemma about a system of reductions (e.g. relations replacing some terms by other terms in a presentation of an algebra). This system of reductions is required to have the following properties:

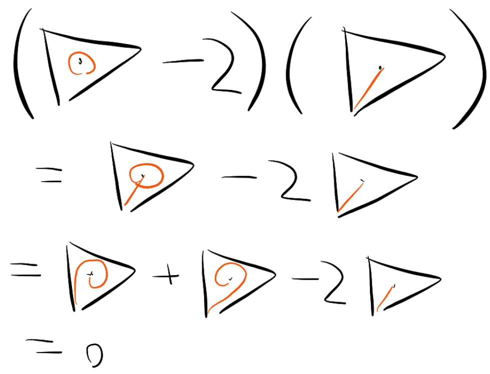

Figure 7: A zero divisor.

1. Local confluence: if R_1, R_2 are two reductions of some x, there is some further chain of reductions which makes them equal.

2. Any chain of reductions terminates.

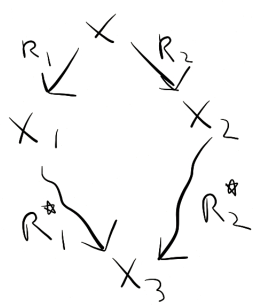

Figure 8: A diamond describing local confluence.

With these hypotheses, any two chains of reductions from x ends at the same place. This proceeds by induction on the length of chains.

These hypotheses are straightforward to verify for the skein relations.

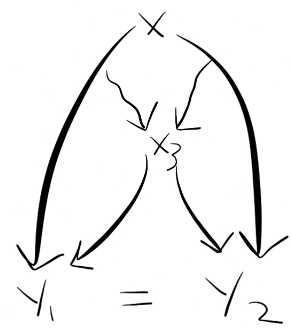

Figure 9: The inductive step.