274 Curves on Surfaces, Lecture 14

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Fall 2012

15 Compactifications, skein theory

Theorem 15.1. *The Thurston compactification is a ball for the following Teichmüller spaces:*

1. $\tilde{\mathcal{T}}_{g,n}$ *(λ-lengths),*
2. $\mathcal{T}_{g,n}$ *(cross-ratios),*
3. $\mathcal{T}_{g,0,n}$ *(cross-ratios).*

It is also a ball for $\mathcal{T}_g$ but this is harder (the coordinates are given by lengths of all simple closed curves).

Proof. Correction: when we discussed the Thurston compactification, we should have embedded not in a projective space but in a sphere (in other words instead of quotienting by nonzero reals we quotient by positive reals). If T denotes a triangulation, we can then measure λ-lengths to get a map $\tilde{\mathcal{T}}_{g,n} \to \mathbb{R}^{|T|}$. Embedding $\mathbb{R}^{|T|}$ into $\mathbb{R}^{|T|+1}$ gives a map to $S^{|T|}$.

Lemma 15.2. *The map* $\overline{\tilde{\mathcal{T}}_{g,n}} \to S^{|T|}$ *above is an embedding.*

Proof. $\tilde{\mathcal{T}}_{g,n}$ embeds into $\mathbb{R}^{|T|}$ due to the existence of the exchange relation: we can express the λ-length of any arc as a Laurent polynomial in the λ-lengths of a fixed triangulation.

To show that the Thurston compactification embeds into $S^{|T|}$, we use the fact that on the new points, our coordinates satisfy a tropical exchange relation, which expresses a new length as a piecewise-linear function of the old lengths. (The key point here is that the original exchange relation is a subtraction-free expression in positive variables. Without this condition, there is no guarantee that a new length is a function of old lengths.) □

The above does not always happen.

Exercise 15.3. *Consider decorated quadrilaterals with three boundary edges of λ-length* 1*. This moduli space embeds into $\mathbb{R}^2$, and the corresponding Thurston compactification does not embed into S^2.*

We conclude that the Thurston compactification of $\tilde{\mathcal{T}}_{g,n}$ is a hemisphere in $S^{|T|}$, hence homeomorphic to a ball. The argument for $\mathcal{T}_{g,0,n}$ is similar. For $\mathcal{T}_{g,n}$, we have additional relations $\prod \tau(E) = 1$ around punctures, which introduce linear relations after taking logarithms. The corresponding closure in $S^{|T|}$ is still homeomorphic to a ball. □

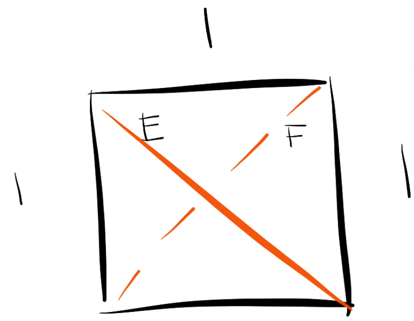


Figure 1: A decorated quadrilateral.

An alternate definition of a lamination, which will hopefully make it more concrete, is the following.

Definition A lamination on $\mathbb{H}^2$ is a collection of pairs of distinct points on the boundary such that the corresponding geodesics do not cross and which is closed as a subset of $(\partial\overline{\mathbb{H}})^2 \setminus \Delta$. A lamination on a hyperbolic surface $\Sigma = \mathbb{H}^2/\Gamma$ is a lamination on $\mathbb{H}^2$ which is invariant under Γ.

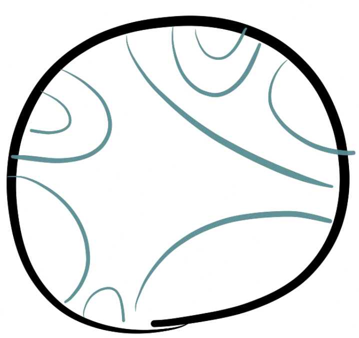


Figure 2: A lamination.

Onto skein theory.

Definition Skein theory for a decorated hyperbolic surface Σ and a multicurve on Σ with endpoints at the punctures is defined up to sign as follows (where ℓ denotes the length of a geodesic representative of a curve, or 0 if there is no such representative):

1. $\lambda(A) = e^{\frac{\ell(A)}{2}}$ for A an arc,
2. $\lambda(C) = 2\cosh\left(\frac{\ell(C)}{2}\right)$ for C a loop,
3. $\lambda(C_1 \cup C_2) = \lambda(C_1)\lambda(C_2)$.

Note that there is no assumption that the multicurve is simple.

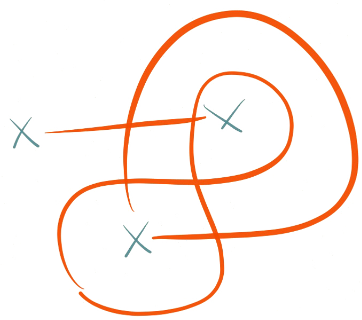
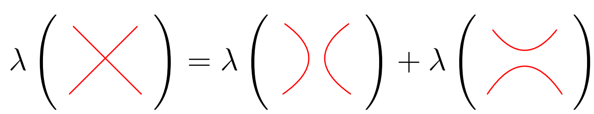


Figure 3: A multicurve, not necessarily simple.

Lemma 15.4. *Up to sign, the skein relation*



(1)

holds.

Some motivation for the above definition is the following.

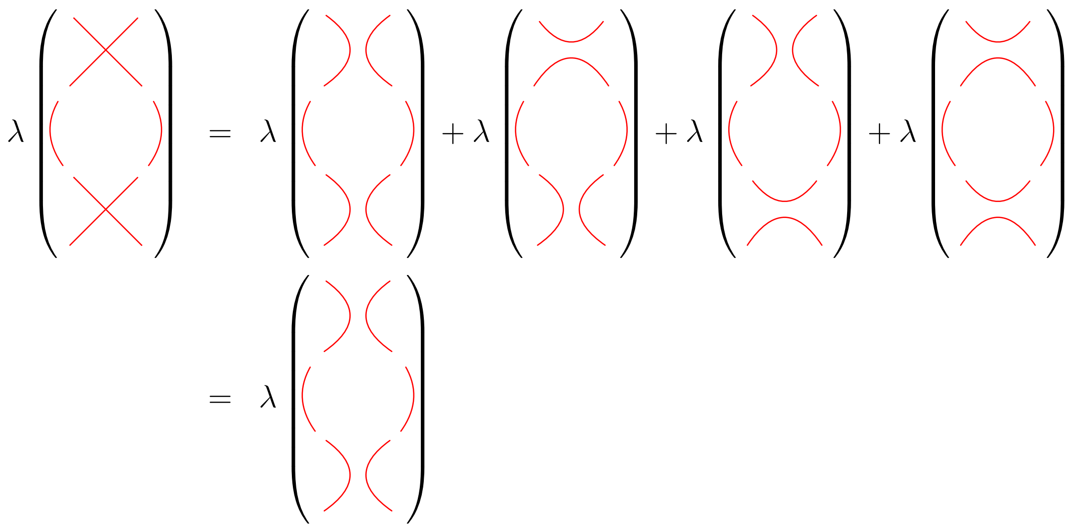
Exercise 15.5. *Let $\phi \in SL_2(\mathbb{R})$ be hyperbolic. Let $\ell(\varphi)$ be the translation length of ϕ (the distance by which ϕ translates on the geodesic between its two fixed points). Then*

$$tr(\phi) = \pm 2cosh\left(\frac{\ell(\varphi)}{2}\right). \tag{2}$$

Definition A *curve diagram* on Σ is an immersion $C \to \Sigma = (S, M)$ (M the set of marked points) where C is a 1-manifold with boundary, the boundary of C is sent to marked points, the interior of C misses marked points, and there are no triple intersections. We consider curve diagrams up to isotopies not changing the crossings. The corresponding set is denoted $\mathrm{CD}(\Sigma)$.

Definition The *skein algebra* $\mathrm{Sk}(\Sigma)$ of Σ is $\mathbb{Z}[\mathrm{CD}(\Sigma)]$ modulo the skein relation, the relation that a closed loop has value -2, and the relation that a loop at a marked point has value 0.

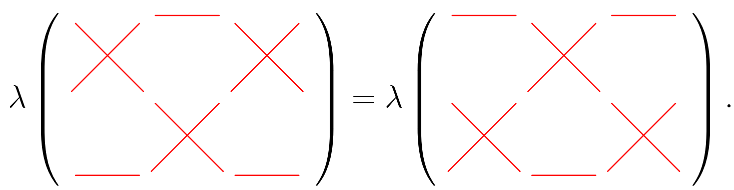
Setting a closed loop to -2 is equivalent to invariance under Reidemeister II:



(3)

(4)

Exercise 15.6. *Check invariance under Reidemeister III. (Do not expand into* 8 *terms.) That is, check that*



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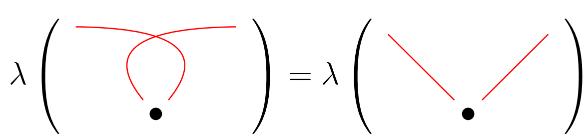
Setting a boundary loop to 0 is equivalent to invariance under a boundary version of Reidemeister II:



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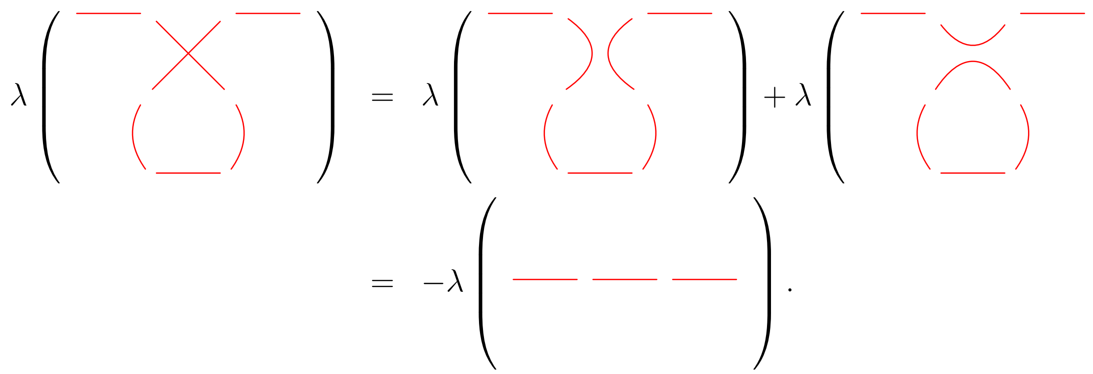
We do not have invariance under Reidemeister I:



(7)

(8)

Instead, adding a twist introduces a sign. In summary, we have the following.

Proposition 15.7. *The class of a curve in Sk*(Σ) *is invariant under RII, RIIb, RIII, and changes by sign under RI.*

Curves up to RII, RIIb, RIII may be regarded as curves up to regular isotopy.

Theorem 15.8. *Simple multicurves (no loops, no boundary loops) form an integral basis for Sk*(Σ)*.*

Proof. There is an expansion map $\mathbb{Z}[\mathrm{CD}(\Sigma)] \to \mathbb{Z}[\mathrm{SC}(\Sigma)]$ (SC the set of simple multicurves) which resolves all crossings using the skein relation and removes all loops and boundary loops. There is also an inclusion map in the other direction, since simple multicurves are curve diagrams. There is also a map $\mathbb{Z}[\mathrm{CD}(\Sigma)] \to \mathrm{Sk}(\Sigma)$ through which the expansion map factors, as well as a map $\mathbb{Z}[\mathrm{SC}(\Sigma)] \to \mathrm{Sk}(\Sigma)$. These maps are inverses. □

There is a map from $\mathrm{Sk}(\Sigma)$ to the upper cluster algebra $\bar{A}(\Sigma)$ associated to Σ. It is not known whether this map is an injection in general. This has been proven under stronger assumptions on the coefficients.

Question from the audience: what does this map look like?

Answer: first we need the following.

Theorem 15.9. *Sk*(Σ) *has a product structure given by union of diagrams.*

Proof. (Sketch) Let $C(\Sigma)$ denote curve diagrams up to regular isotopy (curve diagrams modulo RII, RIIb, RIII). There is a natural map $C(\Sigma) \times C(\Sigma) \to C(\Sigma)$ given by union of diagrams, and this descends to a product structure on $\mathrm{Sk}(\Sigma)$. □

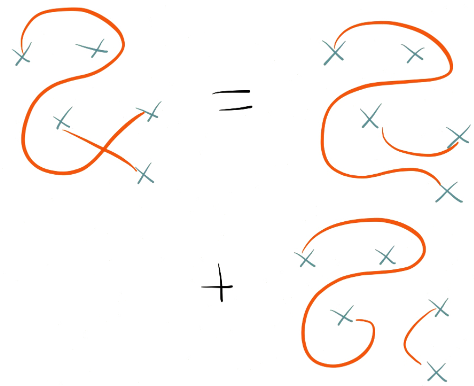


Figure 4: Multiplying by an edge in the triangulation.

Now consider the ring of Laurent polynomials with integer coefficients in the λ-lengths associated to some triangulation T. Given the class of a simple multicurve in $\mathrm{Sk}(\Sigma)$, we can multiply it by an edge in the triangulation and expand using the skein relation.

Repeatedly doing this eventually gives edges in the triangulation, which can be assigned their λ-lengths, and then we can divide by the λ-lengths of the edges we multiplied by.

Exercise 15.10. *Expand a loop in an annulus as a Laurent polynomial.*

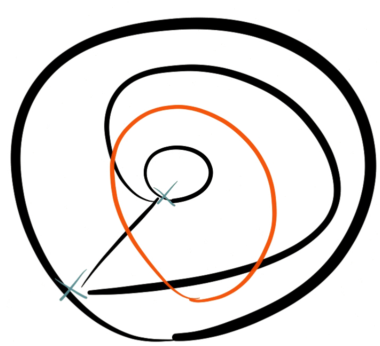


Figure 5: A loop in an annulus.

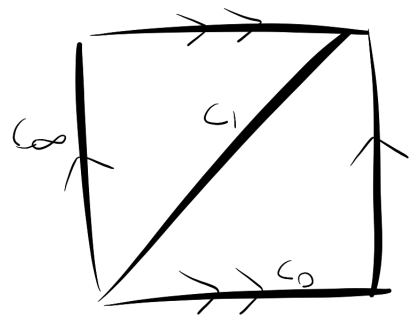


Figure 6: Three curves on a torus.

Exercise 15.11. *Find a relation between the three curves* c_0, c_1, c_∞ *in* $Sk(T^2)$.

Exercise 15.12. *Try to find skein relations for tagged arcs.*