274 Curves on Surfaces, Lecture 12

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13 Laminations and compactifying Teichmüller space

We want to compactify (decorated) Teichmüller space $\tilde{\mathcal{T}}_{g,n}$. One reason we might want to do this is the following: the mapping class group acts on Teichmüller space and we would like to understand the resulting dynamics. However, on a noncompact space this is difficult to do. We have an embedding from decorated Teichmüller space to $\mathbb{R}^I$ given by taking lengths of arcs in a triangulation, but this uses a choice of triangulation and we would like something that doesn't depend on choices so that we can get a natural action of the mapping class group.

We therefore take the lengths of all arcs rather than arcs in a fixed triangulation. This gives now an embedding in an infinite-dimensional Euclidean space. This space in turn embeds into a corresponding projective space, and we define the compactification to be the closure in this projective space. (We can also do this for undecorated Teichmüller space by taking the lengths of all simple closed curves instead of the lengths of arcs in a triangulation, which gives a compactification $\overline{\mathcal{T}}_{g,n}$.) This is the *Thurston compactification*.

The mapping class group still acts on the Thurston compactiication. Moreover:

Theorem 13.1. *$\overline{\mathcal{T}}_{g,n}$ is a ball.*

It follows by the Brouwer fixed point theorem that any element of the mapping class group acting on $\overline{\mathcal{T}}_{g,n}$ has a fixed point.

What happens to the Ptolemy relation in the compactification?

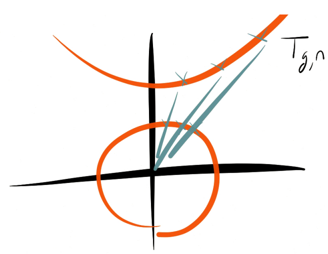

Figure 1: A heuristic picture of a sequence in Teichmüller space heading to infinity in its compactification.

Working in a projective space corresponds to rescaling lengths by some constant:

$$\ell^R(A) = \frac{\ell(A)}{k}. \tag{1}$$

The effect on λ-lengths is

$$\lambda^R(A) = \lambda(A)^{1/k}. \tag{2}$$

The Ptolemy relation now looks like

$$\lambda^R(E)\lambda^R(F) = \lambda^R(A)\lambda^R(C) \oplus_k \lambda^R(B)\lambda^R(D) \tag{3}$$

where

$$x \oplus_k y = \sqrt[k]{x^k + y^k}. \tag{4}$$

As $k \to \infty$, we have

$$\lim_{k\to\infty} x \oplus_k y = \max(x, y). \tag{5}$$

Rewriting in terms of renormalized lengths, we get

$$\ell^R(E) + \ell^R(F) = \max(\ell^R(A) + \ell^R(C), \ell^R(B) + \ell^R(D)). \tag{6}$$

Proposition 13.2. *The above identity holds at all points in compactified Teichmüller space which do not lie in Teichmüller space.*

In other words, compactifying tropicalizes the Ptolemy relation.

Earlier we saw this relation when looking at intersection numbers of curves crossing a triangulation. This suggests that simple closed curves give points in compactified Teichmüller space. We would like to see this geometrically.

Begin with a surface with a hyperbolic structure and a simple closed curve L in that surface. We will construct a sequence of hyperbolic structures going to infinity by placing a long neck where L is.

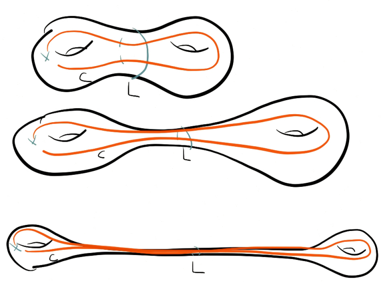

Figure 2: A lengthening neck.

As the neck gets longer, the length of any curve C is dominated by the number of times it crosses L (up to a constant); that is,

$$\ell(C) = k \cdot i(C, L) + O(1). \tag{7}$$

As $k \to \infty$, the lengths of curves approach their intersection numbers in projective space.

More formally, fix a hyperbolic metric Σ_0. Find a geodesic representative for L. Insert a Euclidean cylinder of width t at L. This is enough to get a conformal structure, and then we can uniformize to get a new hyperbolic structure Σ_t.

Example Consider the punctured torus, so the Teichmüller space $\mathcal{T}_{1,1}$. Recall that this can be naturally identified with the hyperbolic plane (which is mildly confusing), and the mapping class group is the usual action of $\mathrm{SL}_2(\mathbb{Z})$ (so the moduli space can be identified with the usual fundamental domain). Punctured tori can be represented as the quotient of $\mathbb{C}$ by the discrete subgroup spanned by two linearly independent $z, w \in \mathbb{C}$.

The boundary of Teichmüller space should be the usual boundary of the hyperbolic plane. Simple closed curves on the torus with rational slope can then be identified with points on the boundary by looking at how inserting cylinders increases the length of the vectors z, w.

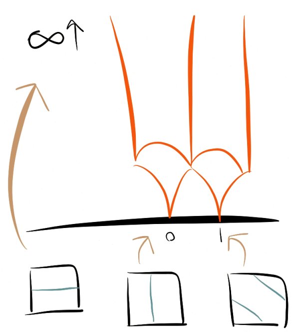

Figure 3: Simple closed curves in a torus and the corresponding boundary points in $\overline{\mathcal{T}}_{1,1}$.

Exercise 13.3. *What simple closed curve corresponds to the limit point at $\frac{3}{5}$?*

We now know that limit points of the compactification of Teichmüller space correspond to tropical solutions to the Ptolemy relations. Integral (rational) limit points correspond to simple closed curves, but this only gives countably many limit points. The real limit points correspond to *laminations*.

More precisely, an *integral decorated lamination* L is a collection of distinct simple closed nontrivial curves with positive weights except that curves surrounding a marked point can have negative weight. These weights describe possible limit behaviors of sequences of decorated hyperbolic structures.

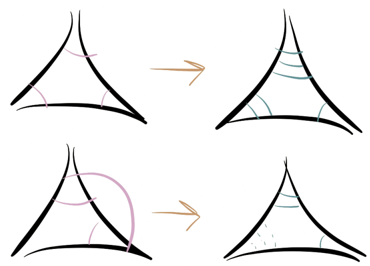

Figure 4: A lamination. The dashed line indicates a negative weight.

We can measure lengths by counting intersection numbers with weights:

$$\ell(A) = i(L, A). \tag{8}$$

Exercise 13.4. *What should the weight of a notched arc be?*

Before we move on to laminations, we should return to the punctured torus with a line of rational slope. $\mathbb{R}^2$ minus the punctures has a hyperbolic metric, and hyperbolically, geodesics will avoid punctures.

More precisely:

Theorem 13.5. *Around each cusp in a hyperbolic surface, simple geodesics do not enter in a horocycle of circumference* 2*.*

In other words, although lines of rational slope look straight in the Euclidean sense, they clump up dramatically in the hyperbolic metric. These clumps look like Cantor sets.

We are now ready to discuss the complete definition of a lamination. A *geodesic lamination* of a hyperbolic surface is a collection of simple geodesics $\{\gamma_\alpha\}$ whose union $\bigcup \gamma_\alpha$ is closed.

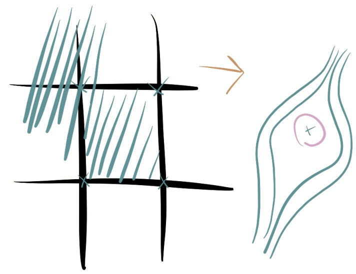

Figure 5: Geodesics avoiding a puncture.

Example Any simple closed curve is a geodesic lamination.

Example Any geodesic that spirals to a closed geodesic (together with the limiting closed geodesic) is a geodesic lamination.

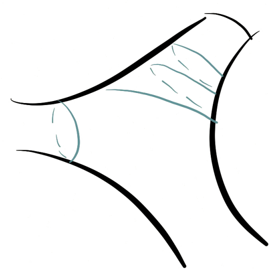

Figure 6: Some atypical examples of laminations.

These are not typical examples. Typically a lamination looks locally like a Cantor set cross the interval.

We want weights as above. A *measured lamination* L is a lamination with a *transverse measure*, namely, a measure on each transverse arc supported on its intersection of L which is transversely invariant in the sense that transverse arcs which are transverse isotopic have the same measure.

Measured laminations give rise to coordinates as follows: we associate to an arc between punctures its total measure.

It is not obvious how to write down measured laminations.

Example A spiraling lamination has no positive measure. Isotoping an arc around the spiral shows that no part of the spiral has positive weight.

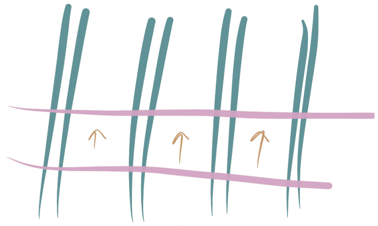

Figure 7: A transverse isotopy between transverse arcs.

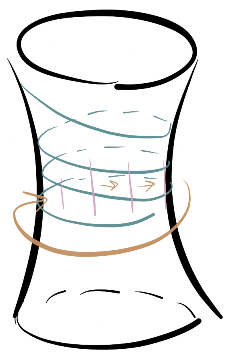

Figure 8: A spiraling lamination.

However, we can write down measured laminations using *train tracks*.

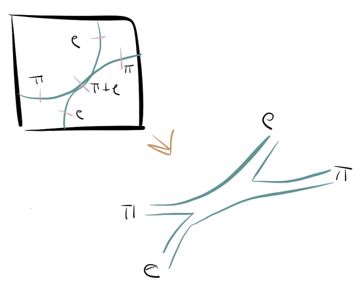

Figure 9: A train track.

Train tracks have an operation defined on them called *splitting*, and by repeatedly splitting a train track we get something approximating a measured lamination.

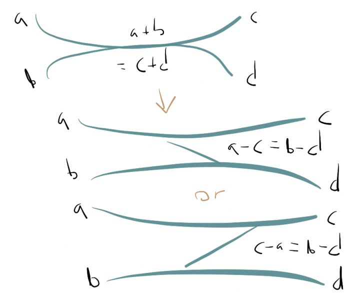

Figure 10: Splitting a train track.