274 Curves on Surfaces, Lecture 10

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11 More about cluster algebras

Last time we discussed conjugate horocycles. This gave a relation $\lambda(A)\lambda(A') = \lambda(B)$ where $\lambda(A')$ is a λ-length measured with respec to the conjugate horocycle. On the other hand, we know that $\ell(h) = \frac{\lambda(B)}{\lambda(A)^2}$, which gives

$$\lambda(A') = \lambda(A)\ell(h) \tag{1}$$

or equivalently taking logarithms,

$$\ell(A') - \ell(A) = 2\ln \ell(h). \tag{2}$$

This can be proven using a scaling argument. The result is clear when $\ell(h) = 1$, since then the horocycle is its own conjugate. In general, a suitable scaling multiplies $\ell(h)$ by c, multiplies $\lambda(A)$ by $\frac{1}{\sqrt{c}}$, and multiplies $\lambda(A')$ by $\sqrt{c}$, so the conclusion follows.

Last time we also asked for a surface giving rise to the affine Dynkin diagrams as quivers. To get $\tilde{A}_{k,\ell}$ we can triangulate an annulus.

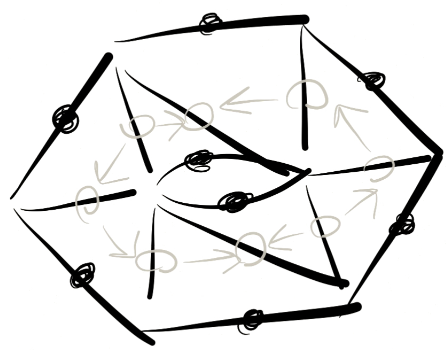

Figure 1: A triangulation giving $\tilde{A}2, 4$.

We also asked for a surface giving rise to D_4 in the orientation where all of the arrows point outward. On the quiver level this can be obtained from the other D_4 we had by mutating twice.

The corresponding geometric exchange relation for the first mutation is

$$x_1 y = x_4 x_3 + x_3 \tag{3}$$

but the actual exchange relation is

$$x_1 x_1' = x_4 + 1. \tag{4}$$

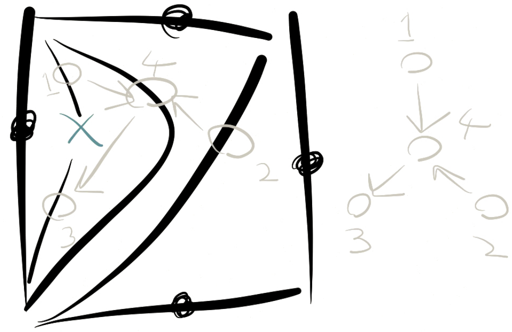

Figure 2: A D_4 with two arrows pointing inward.

As before, this suggests measuring a λ-length with respect to some conjugate horocycle.

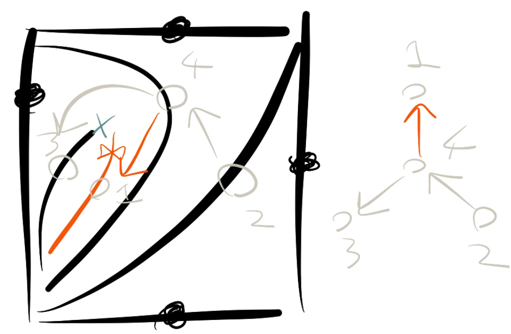

Figure 3: A corrected version of the first mutation giving the correct exchange relation.

Question from the audience: where is the triality symmetry here?

Answer: it appears to be somewhat hidden and is not readily accessible geometrically. Note that quotienting D_4 by triality gives G_2, which is exceptional and does not come from a surface at all.

Another example with hidden symmetry is the 4-punctured sphere. With a tetrahedral triangulation, the corresponding quiver is the octahedron with a certain triangulation. This octahedral quiver can be obtained from a triangulation in a second

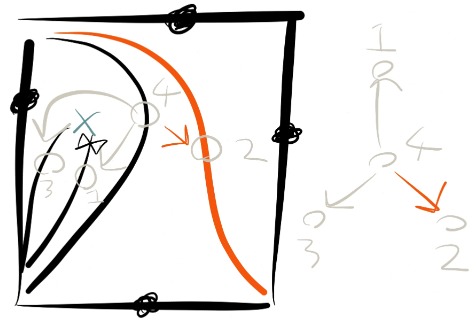

Figure 4: The second mutation.

way, which gives a hidden symmetry (related to Regge symmetry?). More precisely, it can be glued from Type II blocks (see below) in two different ways.

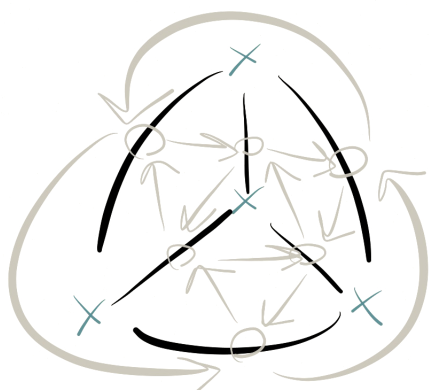

Figure 5: The octahedral quiver.

We will now clarify the geometric meaning of what we have been doing.

A *tagged simple arc* is an arc with one or both ends marked with a notch which does not self-intersect and which does not bound a monogon or a 1-punctured monogon. Notches can only appear at punctures in the interior and should agree at common endpoints if an arc goes from a puncture to itself. Geometrically, a notch indicates that λ-lengths should be measured with respect to the conjugate horocycle. Two tagged arcs are *compatible* if they don't cross and if either

1. the tags agree at common endpoints or

2. the arcs are parallel, one is notched, and one is plain.

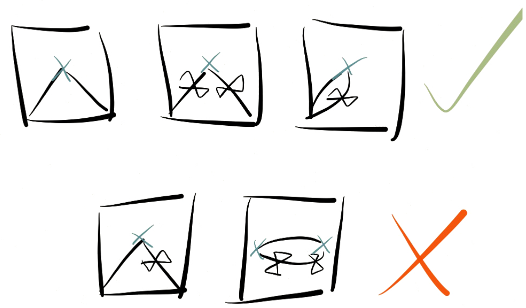

Figure 6: Compatible and incompatible tagged arcs.

A *tagged triangulation* on a surface with a fixed set of marked points is a maximal collection of (distinct) compatible tagged arcs between marked points.

Theorem 11.1. *Any tagged triangulation may be obtained from an ordinary triangulation T by*

1. *replacing self-folded triangles with parallel arcs and*

2. *flipping all tags at some vertices.*

We can construct quivers from a tagged triangulation. The way to remember how this construction works is to remember the relation $\lambda(A)\lambda(A') = \lambda(B)$ for A' a tagged arc parallel to A and B an arc around them. This suggests that when we replace a self-folded triangle with parallel arcs, we effectively double the corresponding vertex in the quiver.

Conversely, to determine when a quiver can come from a tagged triangulation, we can glue *blocks* together (not to themselves) along vertices in such a way that we cancel edges of opposite orientations. Blocks can only be glued along vertices which have not been previously glued.

Any cluster algebra occurring in this way is mutation-finite. However, we don't get some interesting examples, such as the exceptional series.

Exercise 11.2. *Show that it is not possible to obtain E_6, E_7, E_8 by gluing blocks.*

Here is a more precise statement of the classification theorem we stated previously.

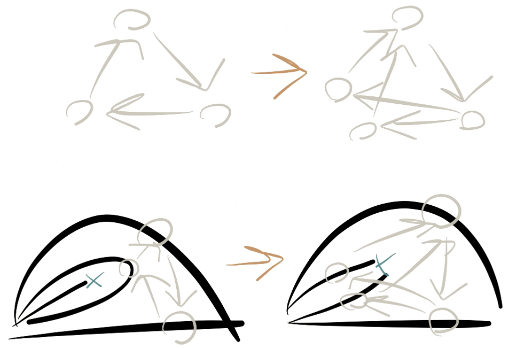

Figure 7: Removing a self-folded triangle and doubling the corresponding vertex.

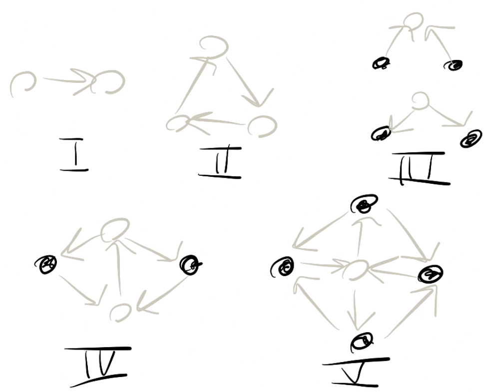

Figure 8: Blocks which glue together to form quivers coming from tagged triangulations.

Theorem 11.3. *Every mutation-finite skew-symmetric cluster algebra is either*

1. *rank* 2*,*
2. *a surface cluster algebra, or*
3. $E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.

It would be interesting to find a better proof of this.

Exercise 11.4. *Where is the default quiver in Bernhard Keller's applet on the above list? Can you mutate it to get to a standard form?*

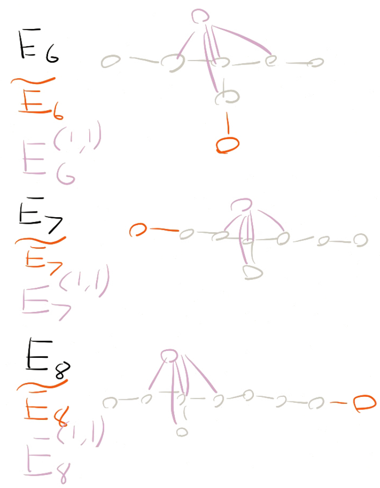

Figure 9: The exceptional diagrams $E_n, \tilde{E}_n$, and $E_n^{(1,1)}$.

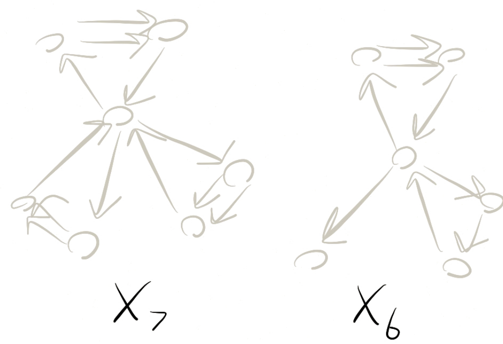

Figure 10: The exceptional diagrams X_7, X_6.

Some of the entries in the above list, such as E_6, E_7, E_8, are not only mutation-finite but of finite type (finitely many cluster variables). The affine ones $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ are not mutation-finite, but the number of clusters reachable after n mutations is $O(n)$ rather than exponential for most quivers.

Exercise 11.5. *Mutate the punctured hexagonal quiver to obtain the* D_6 *quiver.*

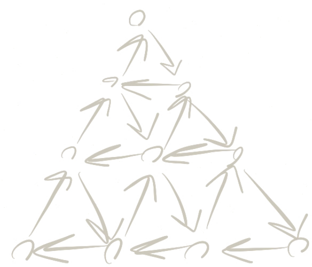

Figure 11: Bernhard Keller's default quiver.

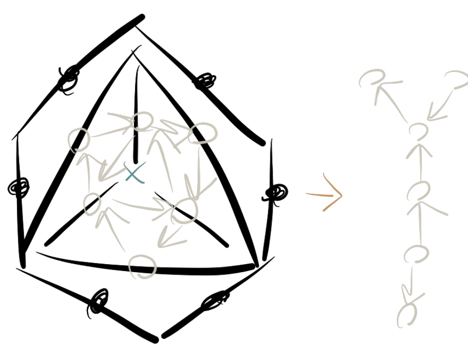

Figure 12: The punctured hexagon and D_6.