274 Curves on Surfaces, Lecture 1

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1 Introduction

We consider simple closed curves on surfaces.

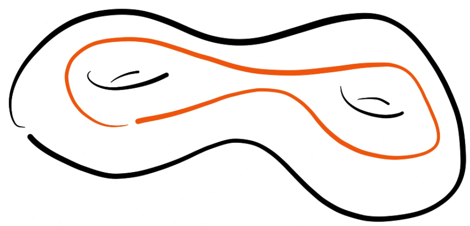

Figure 1: A simple closed curve on a 2-holed torus.

We want to understand such curves. There is a cluster algebra structure related to this, as well as other structures which look like cluster algebras but are not quite formalized. We can study curves on four levels:

1. Tropical - the curves themselves.

2. Algebraic - the cluster algebra. This involves studying Teichmüller space - the space of (uniform) hyperbolic metrics on the surface. Equivalently, the discrete faithful representations of the fundamental group into $\mathrm{PSL}_2(\mathbb{R})$.

3. Quantum - a noncommutative deformation of (the algebra of functions on) Teichmüller space. Related to the Jones polynomial and quantum 3-manifold invariants.

4. Categorical - this does not quite exist yet. Various integers should become objects in a category (and we get the numbers back by taking dimensions, for example). We should get 4-manifold invariants (the last frontier of low-dimensional topology).

The first two levels have been studied for a long time, although there are still open questions. For now we will talk about the first level.

Imagine stirring around the foam in a cup of coffee. The foam may look simple in the beginning but it gradually becomes more complicated and then becomes roughly uniformly distributed in the coffee. More mathematically, consider the disc with three punctures. Begin with a loop around, say, the leftmost two punctures. Permute the punctures and observe what happens to the curve.

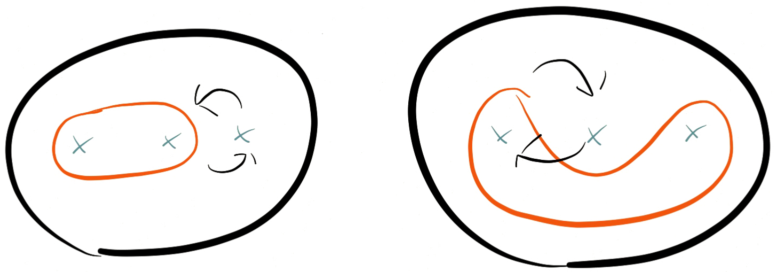

Figure 2: The curve after zero and one permutations.

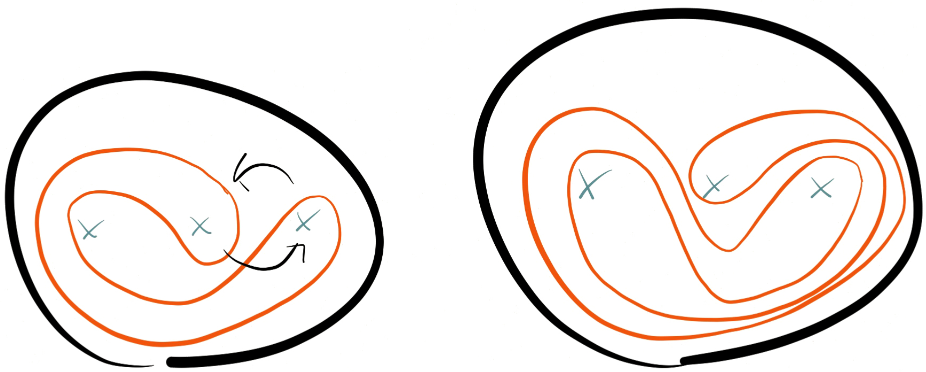

Figure 3: The curve after two and three permutations.

The curve gets more complicated; what can we say about this process?

One way to describe this process is to imagine a movie describing the permutations. The punctures trace out 3-stranded braids, and moving the curve is like trying to pull a rubber band down through the braid. How do we keep track of the curve?

One idea is to triangulate. To do this it will be convenient to add two additional punctures on the boundary. Then the disc can be triangulated by drawing lines between the punctures, and we can keep track of the curve by counting the number of intersections of the curve with the edges of the triangulation.

When we do this we get Fibonacci numbers!

These intersection numbers gives coordinates for simple closed curves up to isotopy which may have multiple components provided that we minimize the number of intersections in a given isotopy class. (These might be called normal curves by analogy with normal surfaces; the condition that the number of intersections is min-

Figure 4: The braid traced out by the punctures over time, along with snapshots of the curve. Down is forwards in time.

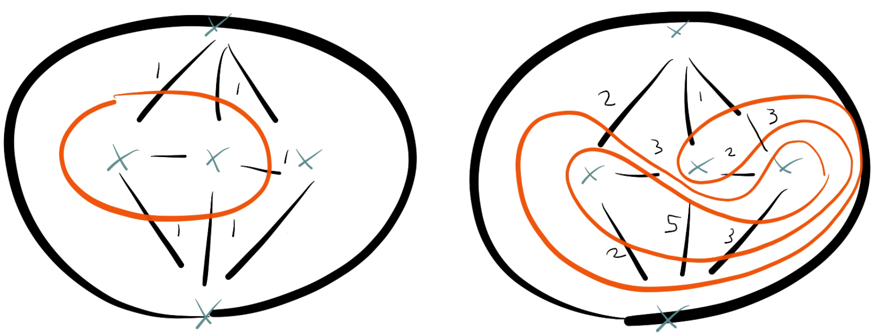

Figure 5: Counts of the number of intersections for zero and three permutations of the curve.

imal is a combinatorial analogue of being a geodesic, and we can find such curves by choosing a Riemannian metric with respect to which the edges of the triangulation are geodesics and choosing geodesic representatives.) This is because, given the number of intersections on the three edges of a triangle, there is a unique way to join them up in such a way that they could form part of a curve if it exists. This is obtained by pairing up intersections near corners.

Exercise 1.1. *Which triples of intersection numbers can be filled in to obtain a curve?*

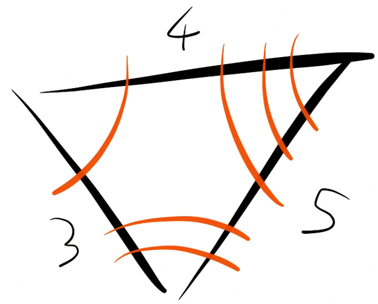

Figure 6: An example of intersection numbers and the corresponding unique curve.

Theorem 1.2. *Coordinates as specified above are unique provided that we restrict to curves with no null-homotopic components.*

One might wonder how general the process we used above to find a triangulation is.

Theorem 1.3. *Every surface with at least one puncture and at least one puncture on each boundary component admits a triangulation in which the punctures are the vertices.*

Triangulation here must be understood in a fairly general sense; for example, the edges of the triangle are allowed to be glued to each other.

Question from the audience: what kind of mathematics is this? How did we know to count the intersections without sign rather than doing the homological thing and counting, say, intersections mod 2?

Answer: there is a braid group B_3 involved which may be thought of as the mapping class group $\mathrm{MCG}(X) = \pi_0(\mathrm{Diff}(X))$ of the disc minus three punctures. (In general B_n may be thought of as either the mapping class group of the n-punctured disc or as $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$.) In this particular case B_3 acts on certain conjugacy classes of π_1 of the disc minus 3 points (the ones which are represented by simple curves). Most conjugacy classes do not have this property. There is a nontrivial theorem we need here:

Theorem 1.4. *(Baer, Epstein, Freedman-Hass-Scott) Two simple curves are isotopic if and only if they are homotopic.*

Does this answer the question? There are a number of motivations for counting intersections without sign (perhaps Dehn was the first one to consider this?). As an application, we will get an algorithm for determining whether an element of the braid group is the identity. Such an element must fix curves around the punctures, and in fact any element which fixes enough curves must be the identity.

We can now examine how the coordinates change when acted on by an element of the mapping class group. If we also act on the triangulation, we will get the same coordinates, but we want to see what the coordinates look like in the old triangulation.

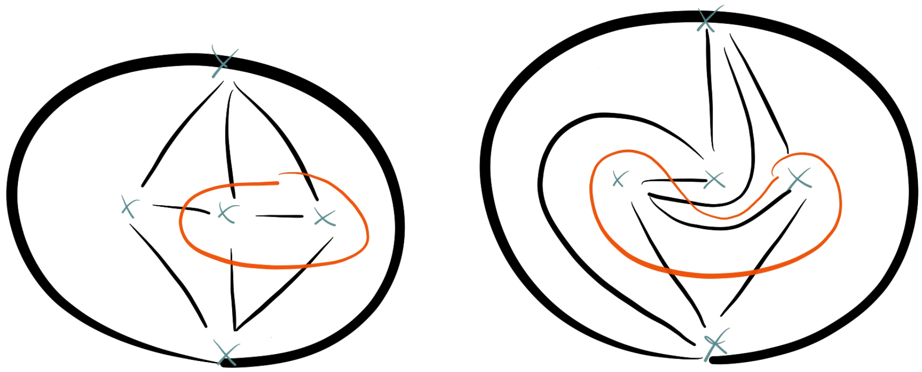

Figure 7: Dragging the triangulation along with the curve.

We can do this by changing the triangulation back. We will do this by flipping the middle edge in a quadrilateral composed of a pair of triangles (mutation?). This operation has many names; it can be interpreted in terms of rotation or the Whitehead move on binary trees.

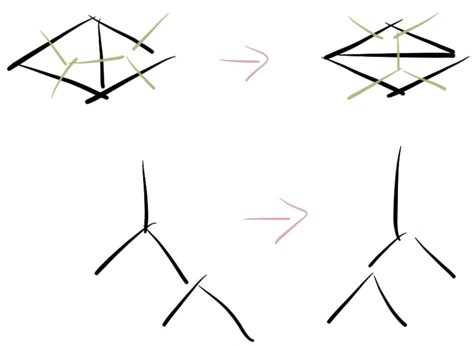

Figure 8: A quadrilateral flip / rotation / Whitehead move.

Theorem 1.5. *Any two triangulations of a surface (with at least two triangles) are related by a sequence of quadrilateral flips.*

It therefore suffices to study how coordinates change under a single flip. The only edge whose intersection number changes is the middle. If the intersection numbers of the outside edges are a, b, c, d and the intersection numbers of the old and new edges are e, f, then in fact

$$e + f = \max(a + c, b + d). \tag{1}$$

(There is a tropical cluster algebra here. This is a version of the Ptolemy relation.)

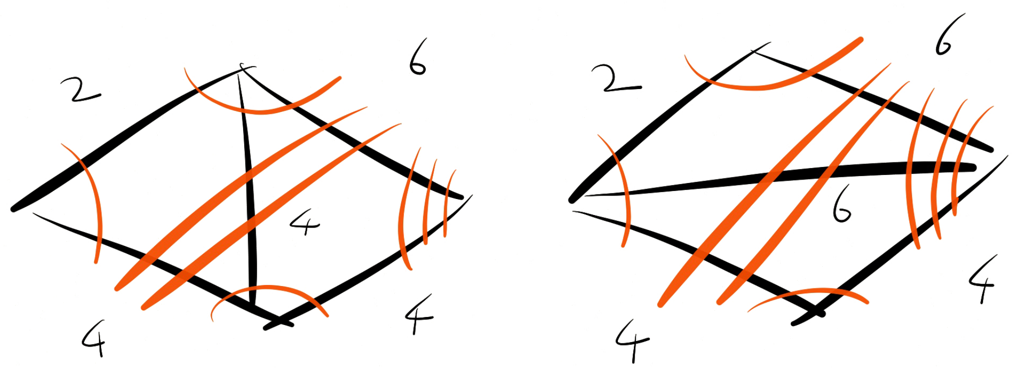

Figure 9: Intersection numbers before and after a quadrilateral flip.

To see this, note that there are six different types of curves: curves between two adjacent edges (four types) and curves crossing opposite sides. The adjacent curves add 1 to $\max(a + c, b + d)$ and to $e + f$ while the non-adjacent curves add 2 to $\max(a+c, b+d)$ and to $e+f$. Which of $a+c$ and $b+d$ is the maximum is determined by which pair of opposite sides is connected by some curve, since this can only occur for one pair.

Exercise 1.6. *Verify that we obtain the Fibonacci numbers in the example. Is there a reason why they appear?*

Exercise 1.7. *What is the asymptotic running time of the braid group algorithm?*

Exercise 1.8. *Choose coordinates for curves in the 3-punctured disc at random in some reasonable sense with some bound. What is the probability that you obtain a single curve? Two curves?*

Possible future topic: the relationship between $e + f = \max(a + c, b + d)$ and normal surfaces in 3-manifolds.