274 Microlocal Geometry, Lecture 9

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9 Calculating intersection cohomology

Let X^{2n} be a ($2n$-dimensional) stratified space with even-dimensional strata. Let $\mathrm{or}_{S^{2n}}$ be the orientation local system on the smooth locus (over $\mathbb{C}$), and let $\mathcal{L}$ be a finite-dimensional local system on S^{2n}.

Theorem 9.1. *There is a unique cochain theory $IC^\bullet(X, \mathcal{L})$ satisfying the following conditions:*

1. *When restricted to the smooth locus, we get cochains $C^\bullet(S^{2n}, \mathcal{L})$ on S^{2n} with values in $\mathcal{L}$;*

2. *Local duality holds in the sense that, for neighborhoods $U_x \ni x$ of a point x with compact closure, we have $IC^\bullet(U_x, \mathcal{L})$ quasi-isomorphic to*

$$IC^{2n-\bullet}\left((U_x, \partial U_x); \mathcal{L}^\vee \otimes or_{S^{2n}}\right)^\vee. \tag{1}$$

3. *Local vanishing holds in the sense that if $x \in S^{2k}$ then the cohomology of $IC^\bullet(U_x, \mathcal{L})$ vanishes for $\bullet \geq n-k$.*

Definition A resolution $\tilde{X} \xrightarrow{\pi} X$ is *small* if, for all $x \in S^{2k} \subseteq X$, where $k < n$, the dimension of the fiber $F_x = \pi^{-1}(x)$ over x is less than $n-k$ (half the codimension).

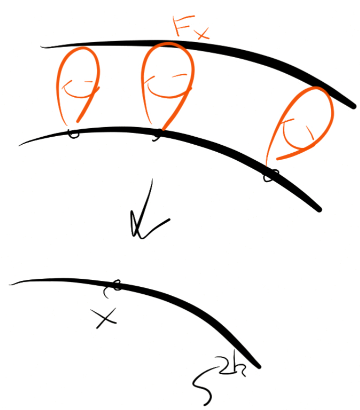

Figure 1: Fibers of a resolution.

Proposition 9.2. *Let $\tilde{X} \to X$ be a small resolution (to be defined later). Then $IC^\bullet(X)$ is quasi-isomorphic to $C^\bullet(\tilde{X})$.*

The definition of small is cooked up more or less to satisfy local vanishing.

Example Let Q^3 be the (complex) 3-dimensional quadric in 2×2 complex matrices such that $\det A = 0$. There is a toric picture of Q as a cone over a square (which is a toric picture of $\mathbb{P}^1 \times \mathbb{P}^1$). There is a small resolution $\tilde{Q}$ consisting of pairs (A, ℓ) where A is a matrix and $\ell \in \ker(A)$ is a line, and a toric picture of $\tilde{Q}$ where the singular point has been resolved to a line. But $\tilde{Q}$ retracts onto $\mathbb{P}^1$ since we can scale the matrix to zero, so the ordinary cohomology of $\tilde{Q}$ is the cohomology of $\mathbb{P}^1$.

Figure 2: A toric picture.

$\mathbb{P}^1 \times \mathbb{P}^1$ arises as follows. If we remove the singular point, we get nonsingular matrices of determinant 0, and these are determined by their column span and their row span up to complex scaling; hence up to complex scaling we get $\mathbb{P}^1 \times \mathbb{P}^1$. If we only work up to real scaling we have a circle action, so the link of the singular point in Q is a circle bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We can then compute the cohomology of the link using the Serre spectral sequence, and from here compute the intersection cohomology of Q. This was harder than finding a resolution.

As a simpler example of the same phenomenon, the quadric

$$Q^2 = \{(x, y, z) \in \mathbb{C}^3 : xz - y^2 = 0\} \tag{2}$$

has singular point $(0, 0, 0)$, and the link of its singular point is a circle bundle over $\mathbb{P}^1$. In fact it is the degree-2 circle bundle, so the link is $\mathbb{RP}^3$. One way to see this is to consider the map

$$\mathbb{C}^2 \ni (a, b) \mapsto (a^2, ab, b^2) \in Q^2 \tag{3}$$

and notice that the link of the singular point in $\mathbb{C}^2$ is S^3 and that the map is a double cover. From here we can compute the intersection cohomology: the cohomology of the link over $\mathbb{C}$ is the cohomology of a 3-sphere, so the intersection cohomology is $\mathbb{C}$ in degree 0 and vanishes otherwise.

(As it turns out, Q^2 does not admit a small resolution, so we cannot use a resolution to compute its intersection cohomology.)

The following theorem is a deep generalization of the hard Lefschetz theorem.

Theorem 9.3. *(Decomposition) Let $\tilde{X} \xrightarrow{\pi} X$ be a proper complex algebraic map. Then there exists a unique collection of closed subvarieties Y of X and local systems $\mathcal{L}_Y$ on their smooth parts such that*

$$IC^*(\tilde{X}) = \bigoplus_{(Y,\mathcal{L}_Y)} IC^*(Y,\mathcal{L}_Y)[shifts]. \tag{4}$$

In a small resolution, the only subvariety that appears is X with the trivial local system.

Question: why might we expect something like this to be true?

Answer: in general, to compute the cohomology of a fibration $\tilde{X} \xrightarrow{\pi} X$ we need a spectral sequence involving the cohomology of X and a fiber. When the spectral sequence is particularly nice the cohomology of $\tilde{X}$ is just the tensor product of the cohomology of X and of a fiber. The decomposition theorem is something like this.

Example We return to the quadric Q^2 and a resolution $\tilde{Q}^2$ of it, with toric pictures as follows.

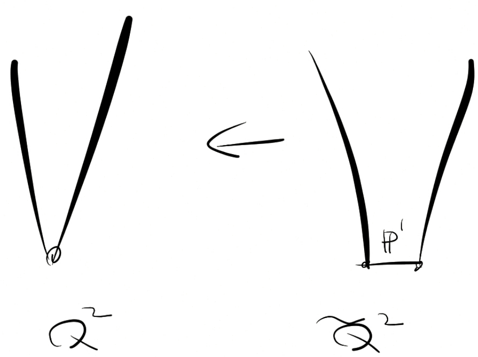

Figure 3: Another toric picture.

Here it turns out that

$$C^\bullet(\tilde{Q}^2) = IC^\bullet(Q^2) \oplus IC^\bullet(\mathrm{pt})[-2]. \tag{5}$$

Exercise 9.4. *Consider the map obtained by collapsing the curve $\mathbb{P}^1 \times \{0\}$ on $\mathbb{P}^1 \times \mathbb{C}$ to a point. This map is not algebraic. Show that the decomposition theorem fails here.*