274 Microlocal Geometry, Lecture 8

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8 More about intersection cohomology

Theorem 8.1. *Let X^{2n} be a (reasonable) $2n$-dimensional stratified space with only even-dimensional strata (e.g. any complex algebraic variety). Write S_{2k} for its $2k$-dimensional strata and assume that S_{2n} is oriented. Then there is a unique cochain theory (sheaf) IC_X such that*

1. *When restricted to S_{2n}, we get cochains on S_{2n} in the usual sense,*
2. *IC_X is self-dual in the sense that $IC_X^\bullet(U_x)$ is quasi-isomorphic to $IC_X^{2n-\bullet}(U_x, \partial U_x)^\vee$ in sufficiently small neighborhoods U_x of points (those looking like cones on the link times a ball),*
3. *IC_X satisfies local vanishing in the sense that if $x \in S_{2k}$ and U_x is a neighborhood as above, then the cohomology of $IC_X^\bullet(U_x)$ vanishes for $\bullet \geq n-k$.*

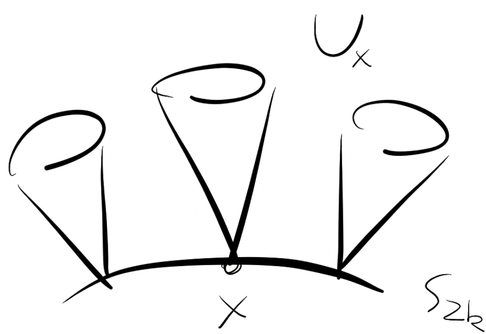

Figure 1: A neighborhood U_x.

Our earlier construction in the special case that $X = \mathrm{Cone}(L^{2n-1})$, where L is $2n-1$-dimensional, satisfied these conditions. The idea is to inductively perform this construction to construct intersection cochains in general.

Example Let $L = S^1$, so X is the cone over a circle. The ordinary cohomology of the link is $\mathbb{C}$ in degree 0 (generated by the whole thing) and 1 in degree 1 (generated by a point). The intersection cohomology is $\mathbb{C}$ in degree 0 (generated by the cone over the link) and zero otherwise, and the relative intersection cohomology is $\mathbb{C}$ in degree 2 (generated by a point in the link).

We now inductively define intersection cochains as follows by inducting on codimension. Let $x \in S_{2k}$. Consider a neighborhood $U_x \ni x$ of the form $\mathrm{Cone}(L_x) \times B^{2k}$. Moving along

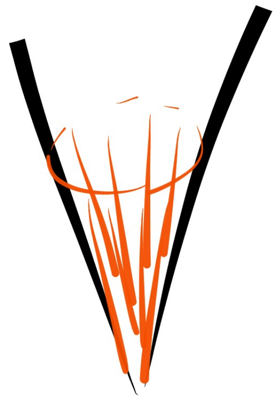

Figure 2: The cone over a circle.

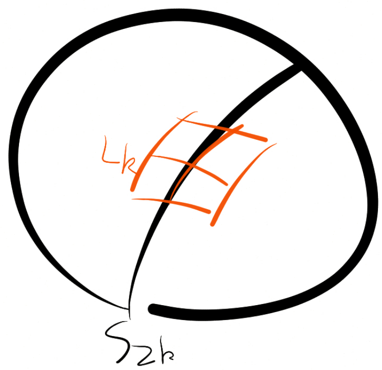

Figure 3: A neighborhood.

S_{2k} won't change anything, so we'll take a slice and reduce to the case that $x \in S_0$ and $U_x \cong \text{Cone}(L_x)$.

(Previously we only allowed L_x to be a manifold, but now L_x is itself a stratified space all of whose strata are odd-dimensional). Since L_x was obtained by taking a slice, by induction we have a self-dual cochain theory on $L_x \times (-\epsilon, \epsilon)$ which restricts (e.g. by taking transverse intersections) to a self-dual cochain theory on L_x. Now we cone off cochains of degree less than or equal to $\frac{\dim L_x - 1}{2}$ as before and ignore the others.

Example Let X be the cone over a pair L of 3-dimensional tori meeting along a circle (so all strata are odd-dimensional, or equivalently have even codimension). L has a finite resolution given by a pair of 3-dimensional tori, so the self-dual cohomology that X sees above is just

$\mathbb{C}^2, \mathbb{C}^6, \mathbb{C}^6, \mathbb{C}^2$. Alternatively, we know what happens on the smooth locus, and the singular locus looks like two tori meeting at a point times an interval.

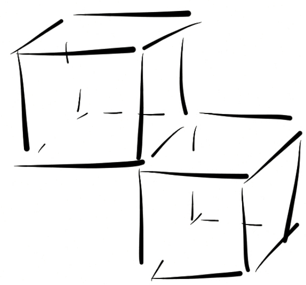

Figure 4: Two 3-tori meeting along a circle.