274 Microlocal Geometry, Lecture 7

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7 Intersection cohomology

Let X be an n-dimensional Whitney stratified space. Assume that X is oriented (by which we mean that the smooth locus is oriented). Because capping with the fundamental class exchanges orientations and coorientations, we may behave as if everything is oriented. Recall also that capping with the fundamental class gave an inclusion $C^k(X) \to C_{n-k}(X)$ of cochains into chains. The difference between these has something to do with the singularities of X. Intersection cohomology tries to find something in between cochains and chains which is self-dual. Roughly speaking, we are trying to find the center term in the sequence of inclusions $L^\infty \to L^2 \to L^1$, or perhaps the sequence of inclusions

$$\text{functions} \to L^2 \to \text{distributions}. \tag{1}$$

(It was conjectured for decades that there is a de Rham presentation of intersection cohomology formalizing this analogy; this has been recently proven?)

An important point of view for us will be that a cochain is completely determined by what it does on the smooth locus. This suggests the possibility of discussing the cohomology of X via the de Rham theory of its smooth locus, with some restrictions.

Example Let X be the kissing banana again.

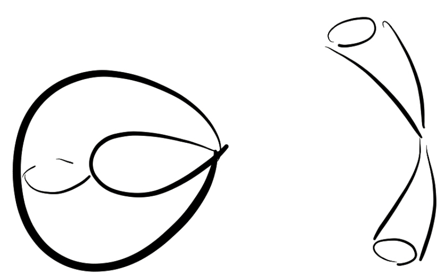

Figure 1: A neighborhood of the singular point.

In a neighborhood of the singular point, the only interesting cochain is the whole thing in degree 0 (it is the only strongly transverse cochain). There are two interesting chains in degree 2 given by the top and bottom half of the neighborhood and an interesting chain in degree 1 given by a line through the singular point.

Intersection cochains are as follows: we keep the top and bottom half of the neighborhood but don't keep the line through the singular point. The corresponding intersection cohomology of X is 1-dimensional in degrees 0 and 2 and 0-dimensional in degree 1; in particular it agrees with the cohomology of the sphere.

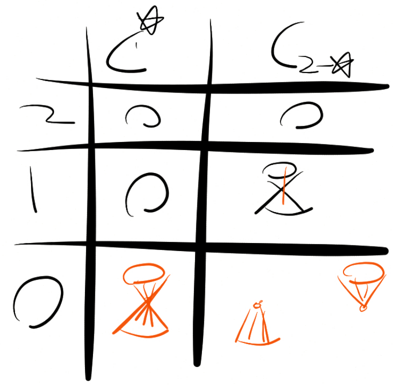

Figure 2: Interesting chains and cochains.

Proposition 7.1. *Let $f : \tilde{X} \to X$ a finite resolution (a finite, in particular proper, map where $\tilde{X}$ is smooth, which is an isomorphism on an open dense subspace). Then $IH^*(X)$, the* intersection cohomology *of X, is the ordinary cohomology of $H^*(\tilde{X})$.*

We can take this as the definition of intersection cohomology. In any case, this tells us what intersection cochains should be.

Note that finiteness is a very strong condition. The kissing banana can be resolved by a sphere, which gives a finite resolution, or by a torus, which does not.

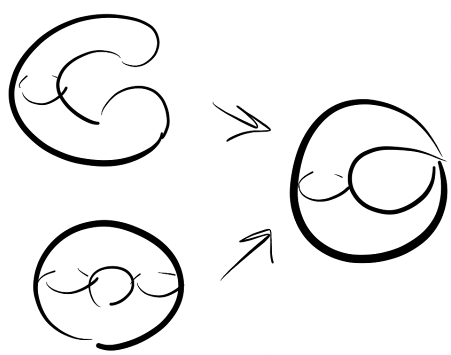

Figure 3: Two resolutions of the kissing banana.

Something funny happens when X has odd-dimensional strata. For example, a wedge of two circles has two finite resolutions which have different cohomology. We will need to restrict to spaces with only even-dimensional strata for uniqueness of the cohomology of resolutions to hold: in particular, any space admitting a stratification by complex manifolds has this property.

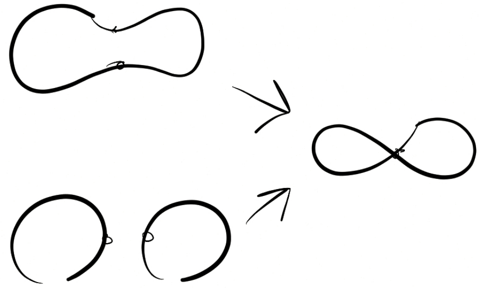

Figure 4: Two finite resolutions of the wedge of two circles.

The local picture of intersection cohomology is as follows. Consider a space X of dimension $2n$ which is the cone over some space Y of odd dimension. Poincaré duality on Y means reflecting over a line between H_{n-1} and H_n. We will allow the cone over anything in codimension $n-1$ and below (dimension n and above) to be a cochain on X.

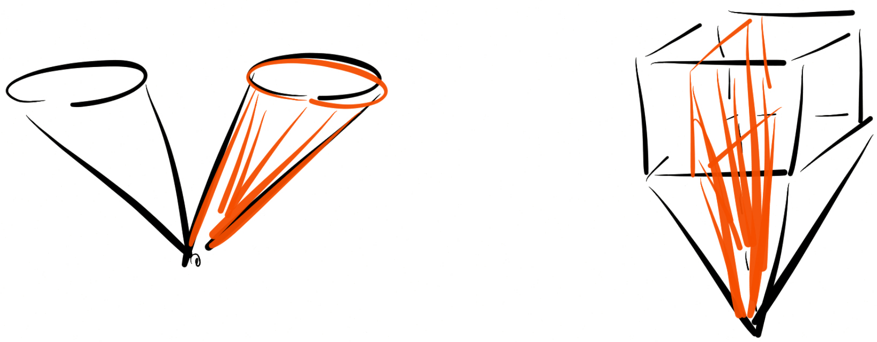

Figure 5: Intersection cochains on cones.

The motivation for the above definition is as follows. For a cone $X = \mathrm{Cone}(Y)$ as above, homology and cohomology are both just $\mathbb{C}$ in degree 0 The failure of Poincaré duality reflects itself in the fact that the relative cohomology of X with respect to Y (which is the reduced cohomology of Y, shifted) should be but is not dual to the cohomology of X; in other words, Lefschetz duality does not hold.

The definition above restores this lack of duality. For a cone $X = \mathrm{Cone}(Y)$, the intersection cohomology $IH^*(X)$ is the cohomology of Y in degrees 0 to $n-1$ and zero otherwise (by taking cones on the low-codimension cochains on Y). The relative intersection cohomology is zero in degrees 0 to n and a shift of the cohomology of Y in degrees $n+1$ to $2n$

(by keeping the high-codimension cochains on Y). Now these two have a Poincaré duality pairing inherited from that of Y.

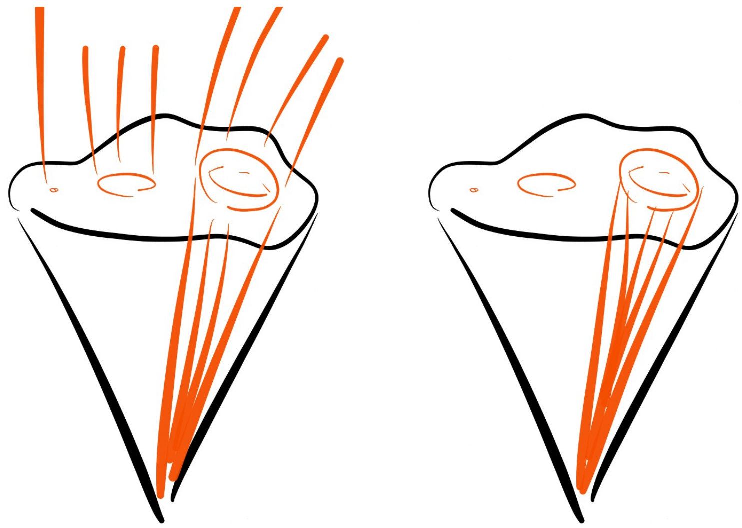

Figure 6: Intersection cochains and relative intersection cochains.

Why did we need X to be even-dimensional? If X is the cone over an even-dimensional Y (say $\dim Y = 2n$), then the middle cohomology $H^n(Y)$ is self-dual, and it's unclear how to divide it up (as we did above). We might want to split $H^n(Y)$ into two dual subspaces, but this is impossible, for example, if its dimension is odd. Even when we can do this, we may not be able to do it canonically.

From the perspective of sheaf theory, this story is complicated. The microlocal viewpoint will make intersection cohomology seem more obvious.