274 Microlocal Geometry, Lecture 6

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6 More about homology and cohomology

Having talked about homology and cohomology for subanalytic spaces, we'd like to talk about long exact sequences, cup and cap products, and Poincaré duality. Eventually we'd like to make our way to intersection cohomology.

We should say some more about strong transversality. With a tube system in place, we can be more specific about what a slice is: namely, it is the inverse image of π_α (part of the data $(\pi_\alpha, E_\alpha, \rho_\alpha, \eta_\alpha)$ associated to a stratum in a tube system).

Why should the construction we've given describe ordinary cohomology? It turns out that we can construct a deformation retract collapsing a neighborhood of the singular stratum to the singular stratum.

Someone wanted to do the Klein bottle as an example.

Example Consider the Klein bottle. The homology computation resembles singular homology and is nothing new. The cohomology computation is more interesting because it does not resemble usual computations with singular or simplicial cohomology. H^2 is $\mathbb{Z}/2\mathbb{Z}$ generated by a point because there is a 1-cochain whose boundary is twice it. H^1 is $\mathbb{Z}$ generated by one of the obvious 1-cochains. H^0 is $\mathbb{Z}$ enerated by the whole thing.

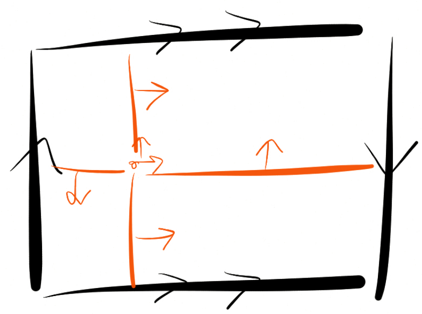

Figure 1: 1-cochains on a Klein bottle.

Example Consider the suspension ΣT^2 of a torus. Here there are two singular points unlike the above. We will compute with $\mathbb{C}$ coefficients. $H_0 = \mathbb{C}$ represented by any point. $H_1 = 0$ because any 1-chain not touching a singular point is the boundary of its cone with respect to the other singular point, and we can always move a 1-chain so that it doesn't touch a singular point. $H_2 = \mathbb{C}^2$ generated by the suspension of the obvious 1-chains in the torus, and $H_3 = \mathbb{C}$ generated by the whole thing.

For cohomology, $H^0 = \mathbb{C}$ generated by the whole thing and $H^1 = 0$ because a codimension-1 cochain can't touch a singular point, so can be coned over. $H^2 = \mathbb{C}^2$ generated by the obvious 1-cochains on the torus because anything they might be the boundary of can't touch the singular points. Similarly $H^3 = \mathbb{C}$ generated by any point.

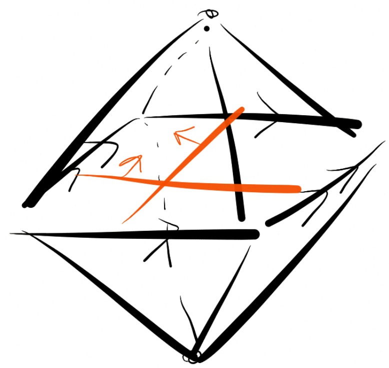

Figure 2: Chains on the suspension of a torus.

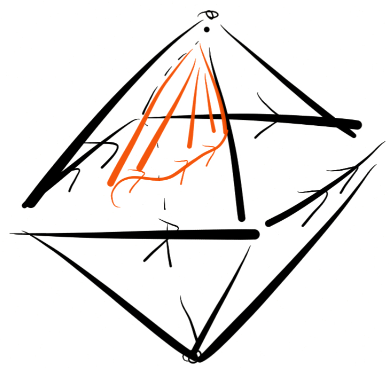

Figure 3: Cochains on the suspension of a torus.

6.1 Functoriality

Let $f : X \to Y$ be a subanalytic map of subanalytic sets. If $\sigma \subseteq X$ is a chain we'd like to map it to $f(\sigma) \subseteq Y$, but the dimension of σ might change. We will fix this by removing any lower-dimensional pieces of $f(\sigma)$.

The same idea doesn't work for pushing forward cochains because we might introduce singularities intersecting the image. Instead, if $\sigma \subseteq Y$ is a cochain we'd like to map it to $f^{-1}(\sigma) \subseteq X$. The problem we might run into is a lack of transversality, which we can fix by moving σ so that it is transverse to f (in the sense that f restricted to each stratum has an image transverse to σ).

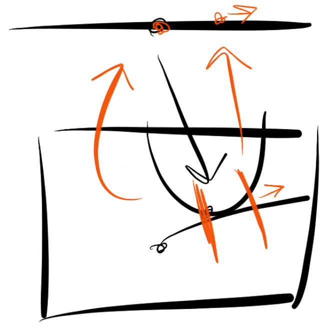

Figure 4: Possible failure of transversality.

6.2 Poincaré duality

Let X be oriented and compact (but possibly with singular points). There is a special element $[x] \in H_n$ called the fundamental class, and taking the cap product with $[x]$ gives an operation $H^k \to H_{n-k}$ given by using the orientation on X to turn a coorientation into an orientation. Because chains, unlike cochains, don't have to worry about singular points, this includes cochains into chains. If X also has no singular points, then cochains can be identified with chains in a way consistent with boundary maps, which reproduces Poincaré duality.

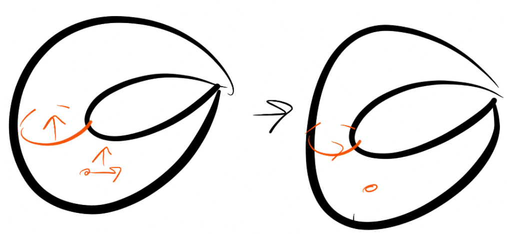

Figure 5: Capping with the fundamental class.

With singularities, the difference between chains and cochains in the above picture is interesting, and there will be a microlocal picture of this.

6.3 Cups and caps

In this picture, the cup product on cohomology is given by taking transverse intersections of cochains. The cap product between homology and cohomology is given in a similar way. The corresponding thing for chains is problematic because it is not always possible to make a chain transverse to itself due to the presence of singularities.

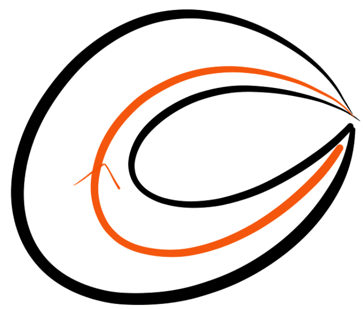

Figure 6: A chain that cannot be made transverse to itself.

6.4 Relative stuff

Let $Y \subseteq X$ be a union of strata in X. We want to define relative chains and cochains to define relative homology and cohomology. Relative chains are chains on X modulo chains on Y. Relative cochains are cochains not intersecting Y. In the following example, relative H_0 vanishes because every point is homologous to a point in Y. Relative H^0 vanishes because no codimension-0 cochain can fail to intersect Y. Relative H^1, H_1 are both $\mathbb{C}^2$.

6.5 Lefschetz duality

Let X be oriented and compact and let $Y \subseteq X$ be closed. Then pairing with the fundamental class gives identifications $H^k(X,Y) \to H_{n-k}(X \setminus Y)$ and $H^k(X \setminus Y) \cong H_{n-k}(X,Y)$. The model we use above makes this straightforward to see.

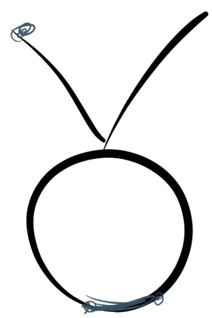

Figure 7: A relative example.