274 Microlocal Geometry, Lecture 5

David Nadler
Notes by Qiaochu Yuan

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5 Homology and cohomology of reasonable spaces

Today we'll only consider subanalytic sets.

Let $X = \mathbb{R}$. Recall that its homology and cohomology are both concentrated in degree 0. But these are very different. We can think of the homology as being generated by a point somewhere on $\mathbb{R}$ with a 1 attached to it. Any two such points are homologous by a chain, which is a line with a number 1 attached to it and an orientation.

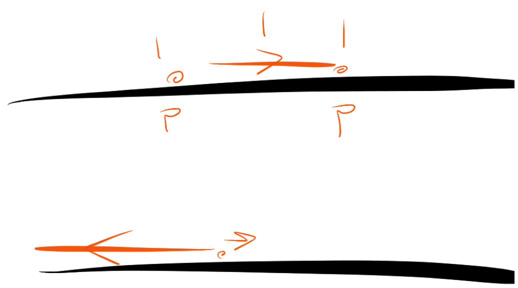

Figure 1: Homology and cohomology.

We can think of the cohomology as being generated by a copy of the entire real line $\mathbb{R}$ with a 1 attached to it (with respect to the singular definition of cohomology, we are implicitly using some kind of Poincarë duality here). H^1 vanishes because any point is the boundary of a cochain given by the line to the left of it.

In general, homology and cohomology linearize a space. We can think of cohomology as being like functions on a space, while homology is like distributions on a space. But in topology everything has to be locally constant.

We'll describe both homology and cohomology via complexes of subanalytic chains and cochains. Here are some guiding principles:

Homology	Cohomology
Compact support	Not necessarily compact
Oriented	Cooriented
Bad intersections with singularities	Strongly transverse to singularities
Indexed by dimension	Indexed by codimension
Covariant	Contravariant
Love relative sets	Hate relative sets

Definition A stratified space has *pure dimension n* if it is the closure of its top stratum, which is n-dimensional.

Definition Let X be a subanalytic set of pure dimension n (closed in some ambient real analytic manifold M). A *subanalytic chain* consists of the following data:

1. a compact subanalytic subset $\sigma_k \subseteq X$ of pure dimension k,
2. an orientation of an open dense smooth locus $\sigma_k^\circ \subseteq \sigma_k$,
3. a number attached to each component of σ_k°.

We can take linear combinations of chains (which has something to do with the numbers and orientations; there's an equivalence relation we need to write down).

Example Let X be a circle decorated with some extra lines. It's homotopy equivalent to a circle, so its homology is generated by an element in degree 0 and an element in degree 1. H_0 can be generated by any point with a 1 attached. Some of these points are singular.

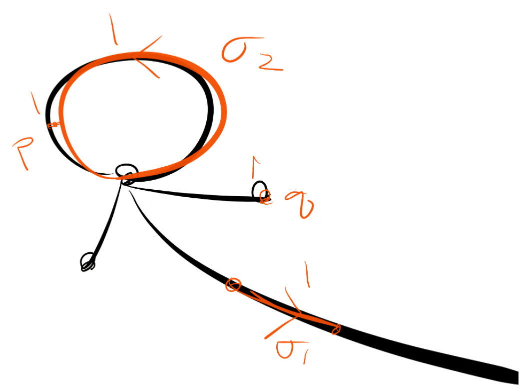

Figure 2: Chains on X.

1-chains are given by unions of line segments (and their endpoints) together with numbers and orientations attached to them. There is a boundary map from 1-chains to 0-chains given by taking the closure of the codimension-1 strata, together with some convention on orientations. H_1 can be generated by the obvious loop, which is stratified by taking a point.

Recall that

1. a *coorientation* of a submanifold in a (Riemannian?) manifold is a nonvanishing section of the top exterior power of the normal bundle,
2. if x is a point in a stratified space with a neighborhood $U_x \cong \mathbb{R}^k \times \text{Cone}(L_x)$, then the *slice* to x is $\text{Cone}(L_x)$,
3. a *singular point* in a stratified space of pure dimension is a point not in the top-dimensional stratum.

Definition Let X be as above. A *subanalytic cochain* consists of the following data:

1. a closed subanalytic subset $\sigma^k \subseteq X$ of pure codimension k which is *strongly transverse* (locally contains the slice to any singular point),
2. a coorientation of an open dense smooth locus $(\sigma^k)^\circ \subseteq \sigma^k$,
3. a number attached to each component of $(\sigma^k)^\circ$.

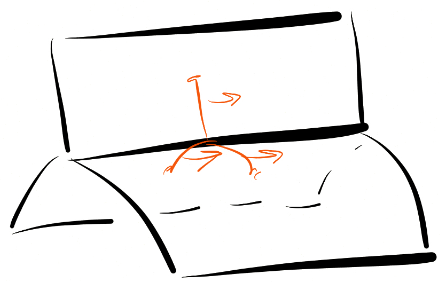

Figure 3: Strong transversality.

Example Let's return to the space X we looked at earlier. A 0-cochain is a line segment with a number attached to it, but coorientations are trivial since line segments have codimension 0. At a singular point, we need to contain a slice. We also have a boundary map from 0-cochains to 1-cochains which is given by intersection with the top-dimensional stratum (this throws away singular points) and then taking the closure of the topological boundary there. H^0 is generated by a cochain covering the entire space.

1-cochains are points, and coorientations are arrows pointing in one direction or another. H^1 is generated by a cochain in the loop; cochains not on the loop are boundaries, so are zero in H^1.

The intuition here is that cochains are like functions in that they should be determined by their behavior on an open dense subset (here the top-dimensional stratum).

Some more words about boundaries. In the following example, the topological boundary of the 0-cochain contains an extra line (the intersection with the line passing through the middle) which we want to get rid of because it is singular in the ambient space. The definition above removes it.

Example Let X be the kissing banana (S^2 with the north and south pole identified). H_0 is generated by a point. For H_1 there are two candidate loops (the space resembles a torus)

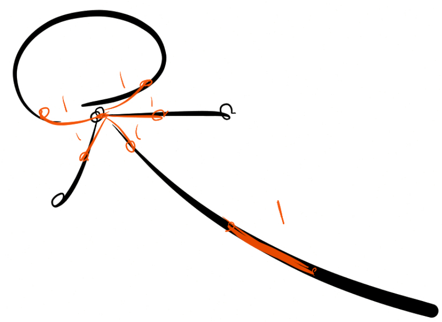

Figure 4: Cochains on X.

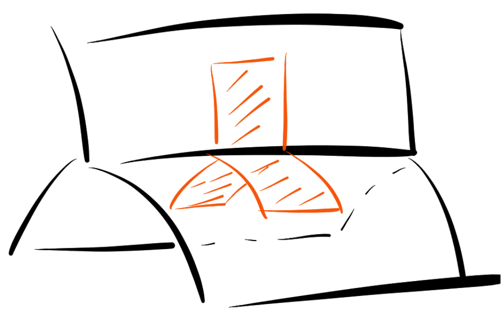

Figure 5: The boundary operator on cochains.

but one of them is zero because it is the boundary of a horn. And H^2 is generated by the entire space; they are all generated by one nonzero element.

Poincarë duality doesn't work here, but it will turn out that H^0, H^1, H^2 are also all generated by one nonzero element. H^0 and H^2 are straightforward, but H^1 is generated by the loop that was zero above; we can't use the horn because it is not transverse to the singular point. The loop that generates H_1 is also not allowed in cohomology because it is not transverse to the singular point.

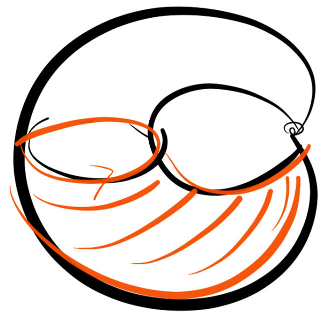

Figure 6: The boundary of the horn is a loop.

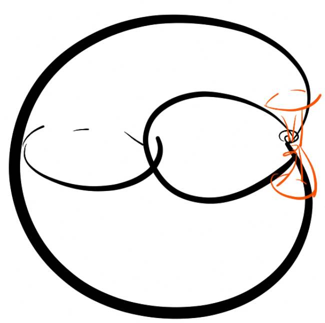

Figure 7: Transversality to the singular point.