274 Microlocal Geometry, Lecture 4

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4 Tame topology

Earlier we described three kinds of reasonable spaces: stratified spaces, tube systems, and Whitney stratifications. The implication between these is that Whitney stratifications give a tube system, and the proof of Thom's first isotopy theorem shows that a tube system gives a stratification.

Theorem 4.1. *A Whitney stratification gives a stratification.*

Proof. (Sketch) Let M be an ambient manifold and M_α a Whitney stratification of it. Pick local coordinates on some M_α so that it looks locally like points of the form $(x_1, ...x_k, 0, ...0)$. Let f be the function $(x_1, ...x_k)$ to $\mathbb{R}^k$. This map turns out to be a submersion on all nearby strata by Whitney's Condition A, so it is a fibration by Thom's theorem. This tells us that small neighborhoods of a point on M_α look like $\mathbb{R}^k$ times the fiber of this map, which we now want to show is a cone. We can do this by taking a suitable neighborhood of a point, removing the point, and applying Thom's theorem to a suitable radial function. □

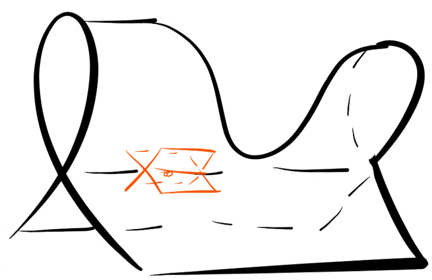

Figure 1: Local coordinates on a stratum.

When do we get Whitney stratifications? One place is when doing real algebraic geometry. Here we study *semialgebraic sets* in $\mathbb{R}^n$, which are described by finite unions and intersections of polynomials equalities and inequalities. The reason we care about inequalities is that sets cut out by polynomial equalities are not closed under projection; for example, the unit circle projects in $\mathbb{R}^2$ projects to a closed interval in $\mathbb{R}$.

Theorem 4.2. *(Tarski-Seidenberg) Semialgebraic sets are closed under projections $\mathbb{R}^{n+1} \to \mathbb{R}^n$.*

This is equivalent to a statement about quantifier elimination, the point being that projecting a set corresponds to tacking on a quantifier.

Theorem 4.3. *All semialgebraic sets admit a Whitney stratification.*

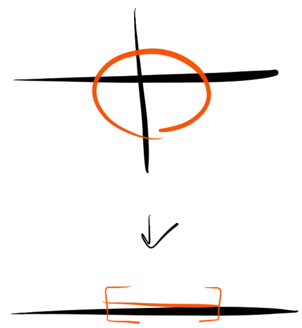

Figure 2: A circle projecting to an interval.

There is a similar theory involving *semianalytic sets*, where we (locally) use real analytic functions rather than polynomials. But the Tarski-Seidenberg theorem fails here even for proper projections.

Example Consider the map

$$f : \mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_1, x_1x_2, x_1x_2e^{x_2}) \in \mathbb{R}^3. \tag{1}$$

Let (y_1, y_2, y_3) be the coordinates on $\mathbb{R}^3$. Ignoring the last coordinate, the first two coordinates almost supply a diffeomorphism, but there is a problem when $x_1 = 0$. When $x_1 = 0$, the y_2 coordinate is identically zero; otherwise everything is fine. So ignoring the third coordinate the image is the y_1, y_2-plane minus the line $y_2 = 0$ together with the point $(y_1, y_2) = (0, 0)$. The actual image is a graph living over this set.

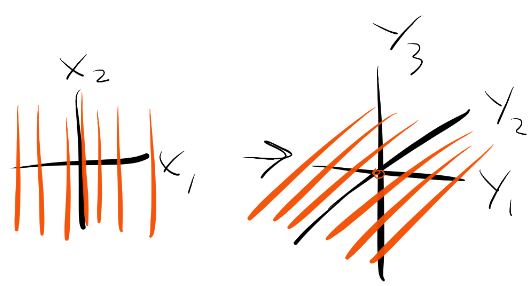

Figure 3: The first two coordinates.

We want to show that no real analytic function can vanish on the actual image without vanishing identically. Suppose $G(x_1, x_1x_2, x_1x_2e^{x_2}) = 0$. Write G as the sum of its homogeneous components

$$G(y_1, y_2, y_3) = \sum_{k=0}^{\infty} G_k(y_1, y_2, y_3). \tag{2}$$

Then

$$G(x_1, x_1x_2, x_1x_2e^{x_2}) = \sum_{k=0}^{\infty} x_1^k G_k(1, x_2, x_2e^{x_2}). \tag{3}$$

If this vanishes identically, then each component $G_k(x_1, x_2, x_2e^{x_2})$ must also vanish identically. But G_k is a polynomial, and e^{x_2} is not an algebraic function.

To fix this, we consider *subanalytic sets*, which are the closure of semianalytic sets under proper images. These also have Whitney stratifications. This all falls under the general heading of analytic-geometric categories: the goal here is to find some reasonable collection of subsets of $\mathbb{R}^n$.

Definition An *analytic-geometric category* is an assignment, to each real analytic manifold M, of a collection $\mathcal{C}(M)$ of subsets of M. This is required to satisfy the following axioms:

1. $\mathcal{C}(M)$ is closed under finite unions, intersections, and complements, and $M \in \mathcal{C}(M)$.

2. The map $A \to A \times \mathbb{R}$ sends subsets in $\mathcal{C}(M)$ to subsets in $\mathcal{C}(M \times \mathbb{R})$.

3. The projection from subsets of M to subsets of N under a real analytic map $f : M \to N$ should send $\mathcal{C}(M)$ to $\mathcal{C}(N)$ provided that the restriction of f to the subset of $\mathcal{C}(M)$ in question is proper.

4. Membership in $\mathcal{C}(M)$ is locally defined: a subset A is in $\mathcal{C}(M)$ iff, in some open cover $M = \bigcup_i U_i$ of M by real analytic submanifolds, $A \cap U_i \in \mathcal{C}(U_i)$.

5. Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

The fifth axiom is the one with real bite. From this data, we can describe a category whose objects are the subsets $A \in \mathcal{C}(M)$ and whose morphisms are functions $f : A \to B$ whose graphs $\Gamma(F) \subseteq A \times B \subseteq M \times N$ are in $\mathcal{C}(M \times N)$.

There is a smallest analytic-geometric category, namely the subanalytic sets. The axioms above are enough to give us some standard tools:

1. Elements of $\mathcal{C}(M)$ are always Whitney stratifiable.

2. Closed sets are the zero loci of functions in the category.

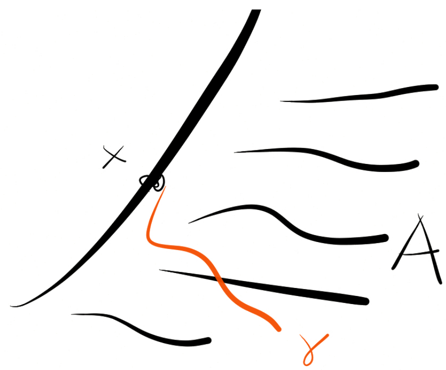

Figure 4: Curve selection.

3. The *curve selection lemma* holds: let $A \in \mathcal{C}(M)$ and let $x \in \text{cl}(A) \setminus A$. Then there exists a curve $\gamma : [0,1) \to M$ in the category such that $\gamma((0,1)) \subset A$ and $\gamma(0) = x$.

The curve selection lemma is used to run arguments such as showing that a sequence of points exhibiting some bad behavior can be lifted to a curve of points exhibiting some bad behavior.

Analytic-geometric categories are the same thing as *o-minimal structures* containing analytic sets, except that we work in $\mathbb{R}^n$ rather than real analytic manifolds.