274 Microlocal Geometry, Lecture 3

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3 Thom's first isotopy theorem

Lemma 3.1. *Let X, Y be Whitney stratified subsets of a manifold M whose strata always intersect transversally. Then the intersections $X_\alpha \cap Y_\beta$ are a Whitney stratification.*

(There may be a problem with local finiteness...?)

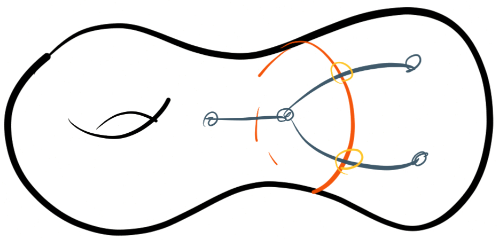

Figure 1: A transverse intersection of stratifications.

In the previous lecture we stated Thom's first isotopy theorem. We can reformulate its hypotheses microlocally. If $f : M \to P$ is a smooth map, microlocally we get a manifold $T^*(P) \times M$ with a projection map to $T^*(P)$ and a derivative map to $T^*(M)$, giving a Lagrangian correspondence between these two symplectic manifolds. It can also be thought of as the conormal bundle $T^*_{\Gamma_f}(M \times P)$ of the graph of f. This is our replacement for f. Recall also that a stratification S_α of $X \subseteq M$ gives a collection of conormal bundles $T^*_{X_\alpha}(M)$ in $T^*(M)$.

We want to rephrase the condition that f is a submersion microlocally. If $P = \mathbb{R}$, then f is a stratified submersion iff

$$f^* \, dt \cap \left(\bigsqcup_\alpha T^*_{X_\alpha}(M) \right) = \emptyset. \tag{1}$$

In general we have the following.

Exercise 3.2. *f is a stratified submersion iff*

$$df^*(p^{-1}(T^*(P) \setminus P)) \cap \left(\bigsqcup_\alpha T^*_{X_\alpha}(M) \right) = \emptyset. \tag{2}$$

The key idea in the proof of Thom's theorem is the notion of a tube system.

Definition A *tube* around a stratum S_α consists of the following data:

1. $E_\alpha \xrightarrow{\pi_\alpha} S_\alpha$, a vector bundle on S_α (of dimension the codimension of S_α, usually the normal bundle);
2. ρ_α, a metric on this vector bundle;
3. η_α, a function $S_\alpha \to \mathbb{R}$;
4. U_α, an open neighborhood of S_α; and
5. a diffeomorphism $U_\alpha \cong \{\rho_\alpha < \eta_\alpha\} \subseteq E_\alpha$.

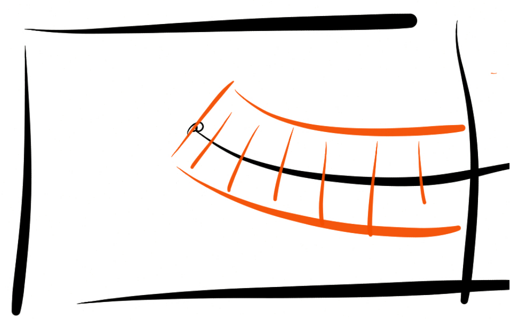

Figure 2: A tube.

We want a collection of tubes around each strata which interact well with each other.

Definition A *tube system* is a collection of tubes around each stratum S_α such that $\pi_\beta = \pi_\beta \pi_\alpha$ (whenever both of these are defined on some U_α) and $\rho_\beta = \rho_\beta \pi_\alpha$ (same condition).

The rest of the proof of Thom's theorem is complicated but has no new ideas: one just applies the proof of Ehresmann's theorem carefully.

Let $f : M \to P$ satisfy the hypotheses of Thom's theorem.

Definition f is *controlled by* a tube system T if $f|_{U_\alpha} = f \circ \pi_\alpha$.

Definition A collection of vector fields v_α, one on each S_α, is *controlled by* a tube system T if $d\pi_\alpha v_\beta = v_\alpha$ and $v_\beta(\rho_\alpha) = 0$ (whenever these make sense).

From here a rough sketch of the proof of Thom's theorem is as follows. Given f, we construct a tube system T controlling f, and we show that vector fields on P lift to controlled vector fields on M. This lets us run the proof of Ehresmann's theorem.

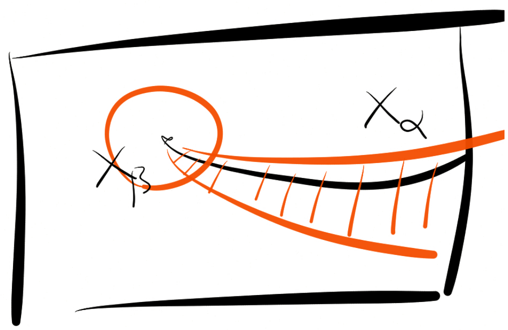

Figure 3: A tube system.