274 Microlocal Geometry, Lecture 26

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26 More about Hamiltonian reduction

Meta-theorem: in practice, every symplectic manifold arises as Hamiltonian reduction of a cotangent bundle (in physics, often an infinite-dimensional vector space).

Last time we considered an action of a Lie group G on a smooth manifold X, giving an induced action by symplectomorphisms on $T^*(X)$. Up to some H^1 business we should get a moment map $H : T^*(X) \to \mathfrak{g}^*$. We can describe this by giving a map $\mathfrak{g} \to \mathcal{O}(T^*(X))$. For starters, the action of G on X gives a map from $\mathfrak{g}$ to vector fields on X. But vector fields are sections of $T(X)$, hence can be evaluated against covectors to get functions on $T^*(X)$ as desired.

When we have a natural moment map H like this it's natural to consider $H^{-1}(0)$ when taking Hamiltonian reduction. This is a union

$$H^{-1}(0) = \bigcup_{G\text{-orbit } Y} T_Y^*(X) \tag{1}$$

of the conormals to the G-orbits, and is in particular singular. Let's call this $T_G^*(X)$. Then the Hamiltonian reduction is $T_G^*(X)/G$. Although they are singular it is possible to describe in what sense they're symplectic using derived geometry.

Example Let $X = \mathbb{CP}^1$ and let $G = \mathbb{C}^\times$ act on it by multiplication on one coordinate. In local coordinates the vector field generating the action above can be written $v = x_1 \, \partial_{x_1}$, so the moment map can be written

$$T^*(X) \ni (x_1, \xi_1) \mapsto x_1 \xi_1 \in \mathfrak{g}^*. \tag{2}$$

The zero level $T_G^*(X)$ is the union of conormals as above; it is a $\mathbb{CP}^1$ with two copies of $\mathbb{C}$ at the two fixed points. The cohomology is $\mathbb{C}, 0, \mathbb{C}$ (the cohomology of the sphere), but the Borel-Moore homology is $0, \mathbb{C}, \mathbb{C}^3$; all 0-cycles can be sent to infinity, there is a 1-cycle connecting the fixed points, and there are three 2-cycles coming from the $\mathbb{CP}^1$ and each of the two $\mathbb{C}$s.

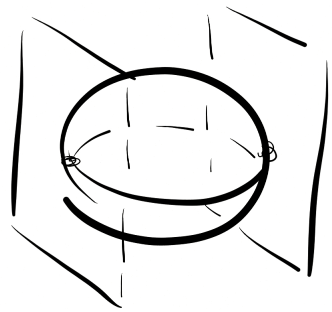

Figure 1: A tie fighter.

Every cochain is in particular a chain, so there is a map from cohomology to Borel-Moore homology. The map in degree 2 is the diagonal $\mathbb{C} \to \mathbb{C}^3$ since the generator of H^2 is the sum of all three 2-cycles we identified above. In particular this map is not an isomorphism, so Poincaré duality is not satisfied. The cone is $0, \mathbb{C}^2, \mathbb{C}^2$. (Recall that way back when we wrote down a sheaf on a space containing a singular fiber X_0 whose sections over X_0 were cohomology, whose global sections were homology, and whose sections away from X_0 were vanishing cycles. We can interpret this cone construction in terms of this sheaf.)

Example Now consider $G = \mathbb{C}^1$ acting on $X = \mathbb{CP}^1$ $(a(x_0 : x_1) = (x_0 : x_1 + ax_0))$. In the coordinate patch where $x_0 = 1$, this is translation. The corresponding vector field is $v = \partial_x$ and the moment map is

$$H(x_1, \xi_1) = \xi_1. \tag{3}$$

In the other coordinate patch the corresponding vector field is $v = \pm x_0^2 \, \partial_{x_0}$ and the moment map is

$$H(x_0, \xi_0) = x_0^2 \xi_0. \tag{4}$$

The zero level has a double line at infinity. The cohomology is again just $\mathbb{C}, 0, \mathbb{C}$, the cohomology of $\mathbb{CP}^1$, since it is a homotopy invariant. The Borel-Moore homology is $0, 0, \mathbb{C}^2$; we've lost a 2-cycle and we don't see the doubling. The cone is $0, \mathbb{C}, \mathbb{C}$.

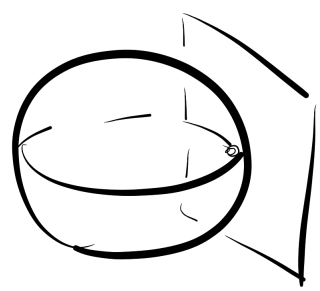

Figure 2: A tie fighter missing one wing.

So, why is all this interesting? There are many natural moduli spaces that are singular and given by equations (e.g. some kind of Hamiltonian reduction), and chains with support conditions on these spaces are important (e.g. in physics). For example, let X be a Riemann surface and consider G-local systems on X. It would be interesting to calculate e.g. microlocal chains with nilpotent support (support on the nilpotent cone in $\mathfrak{g}^*$).