274 Microlocal Geometry, Lecture 25

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25 Hamiltonian reduction

Let (M^{2n}, ω) be a symplectic manifold (so ω is a closed 2-form and $\omega^n \neq 0$). Darboux's theorem assures us that locally M^{2n} is $(\mathbb{R}^{2n}, \omega)$ with

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1}$$

Hence symplectic manifolds in some sense have no local invariants.

Example Let $M = T^*(M)$ be a cotangent bundle. The corresponding symplectic form has the form $\omega = d\theta$ where $\theta \in \Omega^1(T^*(M))$ has the form

$$\theta = \sum y_i \, dx_i. \tag{2}$$

What should a morphism of symplectic manifolds $(M, \omega_M) \to (N, \omega_N)$ be? We can talk about symplectomorphisms, but we really should be talking about Lagrangian correspondences (Lagrangian subvarieties of $(M \times N, -\omega_M \times \omega_N)$; we allow singularities), which are more general (the graph of a symplectomorphism is a Lagrangian correspondence). In particular, a morphism $\text{pt} \to N$ is a Lagrangian submanifold of N and a morphism $M \to \text{pt}$ is a Lagrangian submanifold of M^{op} (M with the opposite symplectic form).

At some point we might want to quotient by a group action $X \to X/G$. Usually the quotient of a symplectic manifold by a group action will fail to be symplectic.

Example Let $M = S^2$ with symplectic form the volume form. (Here every curve is a Lagrangian submanifold.) $G = S^1$ acts by rotation. The quotient map in topological spaces is a line segment; in particular it fails to be even-dimensional and fails to be a manifold.

The singularity is not important; we could take $M = T^2$.

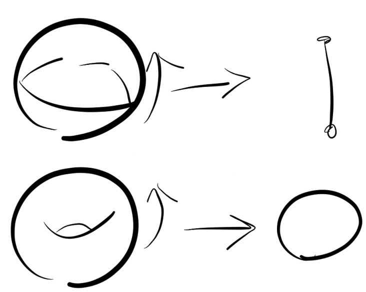

Figure 1: Quotients.

Question: is the problem that S^1 is not symplectic?

Answer: yes and no. In some sense we can replace S^1 with the cotangent bundle $T^*(S^1)$.

The solution to our problems is Hamiltonian reduction. For starters let's think about the quotient by an $\mathbb{R}$-action, so a vector field. The idea is that we won't take a quotient but a subquotient; we'll first pass to a subspace before quotienting. (Correspondences are a good way to talk about subquotients.) Recall that a symplectic form ω gives an identification $\eta : T(M) \to T^*(M)$. If $H : M \to \mathbb{R}$ is a function then it determines a Hamiltonian vector field $\eta^{-1}(dH) = v_H$. (If H is a Hamiltonian of a classical mechanical system then this vector field determines dynamics.) Every Hamiltonian vector field generates a symplectomorphism.

Example Let $M = T^*(\mathbb{R})$ and consider the Hamiltonian

$$H(x, y) = y^2. \tag{3}$$

This Hamiltonian has only a kinetic term. We have $dH = 2y\,dy$ and $v_H = \pm 2y\,dx$ depending on sign conventions. This vector field vanishes when $y = 0$ and looks like a shear in general. This reflects the fact that y is the momentum.

There is an exact sequence

$$0 \to \mathcal{O}_{\text{loc}}(M) \to \mathcal{O}(M) \xrightarrow{d} \Omega^1_{\text{ex}}(M) \tag{4}$$

where $\mathcal{O}_{\text{loc}}(M)$ is the locally constant functions, and a second exact sequence

$$0 \to \Omega^1_{\text{ex}}(M) \xrightarrow{\eta^{-1}} \mathfrak{symp}(M) \xrightarrow{\eta} H^1(M, \mathbb{R}) \to 0 \tag{5}$$

where $\mathfrak{symp}(M)$ is the Lie algebra of vector fields preserving the symplectic form, which can be identified with $\Omega^1_{\text{cl}}(M)$. In particular, if $H^1(M, \mathbb{R})$ vanishes then we can identify $\text{symp}(M)$ with exact 1-forms dH, so an infinitesimal symplectomorphism is the same thing as a function $H : M \to \mathbb{R}$ up to locally constant functions. This information can be thought of as the information of a function $H : M \to \mathfrak{g}^*$ where $\mathfrak{g} = \mathbb{R}$ is the Lie algebra of $\mathbb{R}$. This is a moment map, and we will perform Hamiltonian reduction by first restricting to a fixed value of the moment map and then quotienting:

$$(M//\mathbb{R})_\lambda = H^{-1}(\lambda)/\mathbb{R}. \tag{6}$$

Example When $M = S^2$ we can take H to be a coordinate, and then the Hamiltonian reduction is either a point or empty.

Exercise 25.1. *If λ is a regular value (and maybe the action needs to be reasonable) then the Hamiltonian reduction is symplectic.*

This gives a correspondence $M \leftarrow H^{-1}(\lambda) \to (M//\mathbb{R})_\lambda$.

In general, a moment map for a group action of a Lie group G on a symplectic manifold M is a map

$$H : M \to \mathfrak{g}^* \tag{7}$$

which is G-equivariant and which has the property that, for $v \in \mathfrak{g}$, the image of $v \in \mathfrak{symp}(M)$ is $\eta^{-1}(d(H(v)))$.

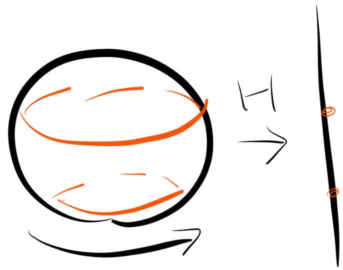

Figure 2: Hamiltonian reduction.

Example Let $G = \mathrm{SU}(2)$ acting on S^2. Here the moment map is a map from S^2 to a 3-dimensional vector space which in good cases is an inclusion.

Exercise 25.2. *Suppose G acts on a manifold X. Find the moment map for G acting on $T^*(X)$. In particular do this in the case that $X = G/B$.*

Moment maps are a good source of interesting maps with interesting fibers which we can apply the machinery we've developed to.