274 Microlocal Geometry, Lecture 24

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Fall 2013

24 Microlocal homology

Recall that we started with a map $f : X \to Y \cong \mathbb{C}$ and a sheaf $F \in \mathrm{Sh}(X)$, replaced it with the projection map $\pi_Y : X \times Y \to Y$ and a sheaf $\tilde{F} = (\Gamma_f)_* F \in \mathrm{Sh}(X \times Y)$, then deformed to the normal cone to get the projection map $\pi : N_{X/(X\times Y)} \to N_{0/Y}$ and a sheaf $\psi\tilde{F} \in \mathrm{Sh}(N_{X/(X\times Y)})$. This is a kind of universal nearby cycles.

We would like to add one more step to this procedure, namely a Fourier transform, which will give us the projection map $\pi : N^*_{X/(X\times Y)} \to N^*_{0/Y}$ and a sheaf $\hat{F} \in \mathrm{Sh}(N^*_{X/(X\times Y)})$. The Fourier transform $\hat{F}$ has the property that its restriction to the zero fiber will be the horizontally compactly supported sections of $\psi\tilde{F}$ and its restriction to the ε-fiber will be vanishing cycles $\varphi_\varepsilon F$ (all possibly up to shifts).

In particular, suppose $F = C^\bullet_X$. Then $\hat{F}$ is supported on $X_0 \times \mathbb{C}$. Its sections over $X_0 \times \{0\}$ are $C^\bullet_{X_0}$ (cohomology), while its sections over $X_0 \times \mathbb{C}$ are Borel-Moore chains $C^{X_0}_{\mathrm{BM}}$ (homology). There is a natural map between these given by capping with the fundamental class.

Here is a thesis problem: $\hat{F}$ can be restricted to various subvarieties. These restrictions give chain theories living between homology and cohomology. For example, in Hamiltonian reduction we could take f to be a moment map. Then the sections over $X_0 \times \mathfrak{g}^*$ would be homology, the sections over $X_0 \times \{0\}$ would be cohomology, but we could also consider, say, the sections over $X_0 \times N$ where N is the nilpotent cone. What do these sections look like?

In the one-dimensional case, for example, we can consider sections over $X_0 \times \mathbb{C}^\times$, which gives the cone of $1 - m$ where m is the monodromy on φ.

Example Consider again

$$f : \mathbb{C} \cong X \ni z \mapsto z^2 \in Y \cong \mathbb{C}. \tag{1}$$

After deformation to the normal cone, $\psi\tilde{F} \in \mathrm{Sh}(N)$, where $N = N_{0/Y}$, is a direct sum $C^\bullet_N \oplus j_* L_{-1}$ where j is the inclusion $j : N \setminus \{0\} \to N$ and L_{-1} is the local system with monodromy -1.

Now we perform a Fourier transform. Recall that this exchanges skyscraper sheaves and constant sheaves. We get the sheaf $C^\bullet_{\{0\}} \oplus j_* L_{-1}$ (the second direct summand is Fourier self-dual). Here capping with the fundamental class is an isomorphism, so Poincaré duality is satisfied (rationally; integrally there will be problems).

Recall that the Fourier transform is a functor $\mathrm{Sh}(E) \to \mathrm{Sh}(E^*)$ where E is a vector bundle on some space X. The set of pairs $k = \{(v, \lambda) \in E \times E^* : \lambda(v) \geq 0\}$ gives rise to a sheaf $K = C^\bullet_k$ on the bundle product $E \times E^*$, and the Fourier transform is

$$\mathrm{FT}(F) = (\pi_{E^*})_! \left(K \otimes \pi_E^* F\right). \tag{2}$$

This is an equivalence from $\mathbb{R}_+$-conical sheaves on E to $\mathbb{R}_+$-conical sheaves on E^*.

Now let's do another example.

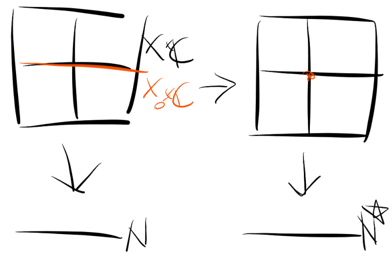

Figure 1: Fourier transform.

Example Again consider

$$f : \mathbb{C}^2 \cong X \ni (x, y) \mapsto xy \in Y \cong \mathbb{C} \tag{3}$$

and the constant sheaf. After deforming to the normal cone and computing universal nearby cycles, the resulting sheaf is supported on $X_0 \times N$, equipped with the projection to N. The Fourier transform is supported on axes; it vanishes most of the time because there aren't vanishing cycles most of the time.

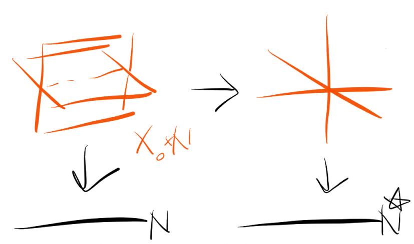

Figure 2: Another Fourier transform.

We get a sheaf whose global sections are Borel-Moore chains on the special fiber X_0 (a cone) and whose sections supported on X_0 is the constant sheaf. The former is $\mathbb{C}^2$ in degree 0 and $\mathbb{C}$ in degree 1 while the latter is $\mathbb{C}$ in degree 0, so Poincaré duality is not satisfied. The cone of the map from the latter to the former, which is the diagonal map $\mathbb{C} \to \mathbb{C}^2$ in degree 0, is $\mathbb{C}$ in degrees 0 and 1. This is the cone of $1 - m$ where m is the monodromy on φ. By contrast, in the previous example the cone of $1 - m$ was trivial.

Some philosophical points.

One lesson here is that homology of a space (say a local complete intersection, realized as a singular fiber as above) can be spread out in cotangent directions, and also that the difference between homology and cohomology on such a fiber has something to do with vanishing cycles.

The sheaf $\hat{F}$ we produced is a perverse sheaf. The $\mathfrak{G}_m$-action gives a natural filtration on it that has something to do with intersection cohomology.