274 Microlocal Geometry, Lecture 23

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23 More about deformation to the normal cone

Example Consider the map

$$f : X \ni z \mapsto z^n \in Y \tag{1}$$

where $X = Y = \mathbb{C}$, and the constant sheaf $F = \mathbb{C}_X$ on X. We already know that the nearby cycles are $\mathbb{C}^n$ and the vanishing cycles are $\mathbb{C}^{n-1}$. But we want to try deformation to the normal cone here. So now we consider the sheaf $\tilde{F} = \mathbb{C}_{\Gamma_f}$ supported on $\Gamma_f \subseteq X \times Y$ and the projection $\pi_Y : X \times Y \to Y$.

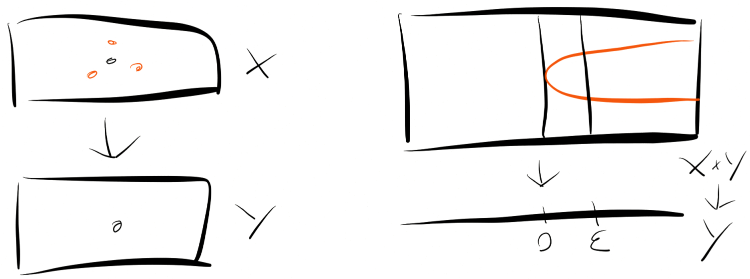

Figure 1: A complex picture and a real picture.

Here the normal bundle $N_{X/(X\times Y)}$ is trivial. When we degenerate to the normal cone the graph $y - x^n = 0$ starts to look $y - (\text{big})x^n = 0$ and eventually becomes $x^n = 0$. The new sheaf $\psi\tilde{F}$ on $N_{X/(X\times Y)}$ is $\mathbb{C}^n$ on the degenerate graph, $\mathbb{C}$ on the singular point, with monodromy around the singular point, and the restriction map $\mathbb{C}^n \to \mathbb{C}$ is sum. In particular, it has nicer support than before. This reproduces the nearby and vanishing calculations we did earlier.

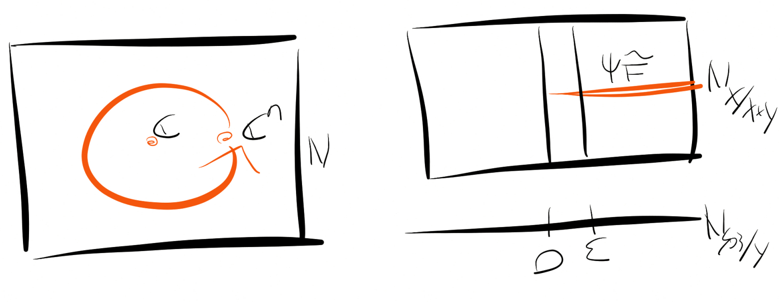

Figure 2: Monodromy and support.

The goal of doing this is not to make calculations easier; rather it is to talk about Fourier transforms, which we need to linearize to do first.

Example Consider the map

$$f : X \cong \mathbb{C}^2 \ni (x_1, x_2) \mapsto x_1 x_2 \in \mathbb{C} \cong Y \tag{2}$$

and F the constant sheaf again. Once again we can consider the sheaf $\tilde{F}$ supported on the graph $\Gamma_f \subset X \times Y \cong \mathbb{C}^3$ and the projection $\pi_Y : X \times Y \to Y$.

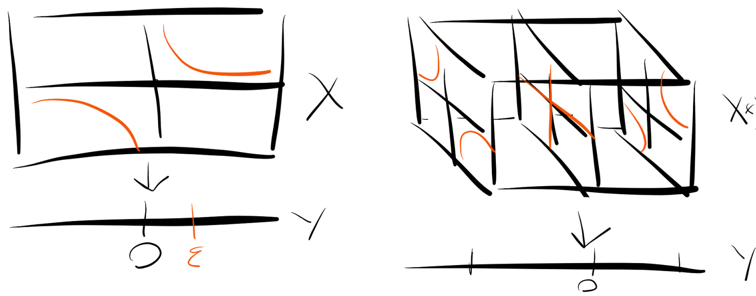

Figure 3: A real picture of f and the corresponding projection.

The degeneration to the normal cone sends the graph $y = x_1 x_2$ to the family of graphs $ty = x_1 x_2$ as $t \to \infty$. Before we pictured a parabola getting thin; now we should picture a saddle getting steep and thin. On each slice, the corresponding hyperbolas are getting thin. The support of $\psi \tilde{F}$ is now $N_{X/(X \times Y)}$ restricted to X_0. We again reproduce the nearby and vanishing cycles computations from earlier.

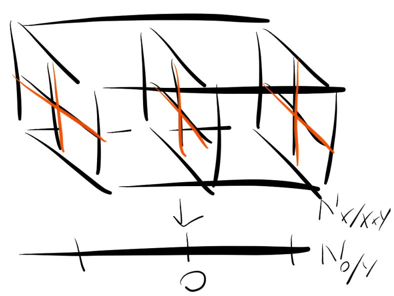

Figure 4: Thin hyperbolas.

One way to describe what we're doing is that we're computing nearby cycles for all ε simultaneously.

Example For a new example, consider

$$f : X \cong \mathbb{C}^3 \ni (x, y, z) \mapsto xy - z^2 \in \mathbb{C} \cong Y. \tag{3}$$

We can think of this as the determinant map $\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$. This is a 2-dimensional quadric; we already computed the 0-dimensional and 1-dimensional quadrics. The 1-dimensional case was somewhat complicated but the 0-dimensional and 2-dimensional case are complicated; the complication oscillates based on parity. For k-dimensional quadrics the nearby slice will be diffeomorphic to $T(S^k)$, which reflects that S^k is the vanishing cycle. (When $k = 0$ this is two points and when $k = 1$ this is a cylinder.)

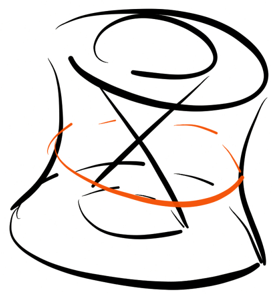

Figure 5: The vanishing cycle.

For $k = 2$ we claim that ψC_X is $\mathbb{C}_{X_0} \oplus \mathbb{C}_{\{0\}}[-2]$, which is simpler than the $k = 1$ case.

To write down this computation let's recall that we resolved $\mathfrak{sl}_2(\mathbb{C})$ to

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \{(A, \ell) \in \mathfrak{sl}_2(\mathbb{C}) \times \mathbb{P}^1 | A\ell = \ell\} \tag{4}$$

which is equipped with a map to $\mathfrak{sl}_2(\mathbb{C})$. The generic fiber has 2 points since a generic matrix has 2 eigenvectors, but the special fiber has been blown up. This is called the Grothendieck-Springer resolution.

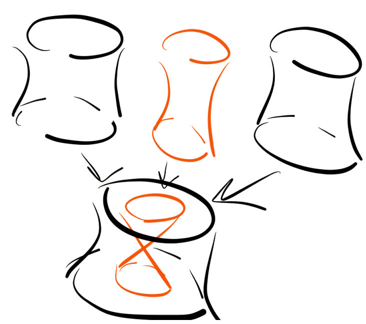

Figure 6: The resolution.

Nearby cycles commutes with proper pushforward, so we can compute nearby cycles on $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, where nothing happens because the family is topologically trivial, and then compute the pushforward by the decomposition theorem.