274 Microlocal Geometry, Lecture 22

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22 Deformation to the normal cone

Last time we gave two definitions of nearby cycles. One was in terms of an almost retract $\pi : X \to X_0$ to the singular fiber. The other was in terms of pulling back and pushing forward from the universal cover.

Question: in general, let $f : \tilde{B} \to B$ be, say, a fibration. How should I think about $f_* f^* F$?

Answer: there's a projection formula which in this case shows that

$$f_* f^* F \cong f_* \mathbb{C}_{\tilde{B}} \otimes F. \tag{1}$$

As another example, if f is a covering map and F is a local system then it is just a representation of $\pi_1(B)$, the pullback is the restriction to $\pi_1(\tilde{B})$, and the pushforward is the induction.

Now let's talk about deformation to the normal cone. Recall that by considering the graph map $\Gamma_f : X \to X \times \mathbb{C}$ we reduced the study of nearby and vanishing cycles to the case of a projection map $\pi : X \times \mathbb{C} \to \mathbb{C}$ at the cost of considering arbitrary sheaves. We can try to make our lives easier by simplifying the sheaf by degenerating to the normal cone of $X \times \{0\} \subset X \times \mathbb{C}$.

In general, let $Y \subset M$ be a submanifold of a manifold. Recall that the normal bundle $N_{Y/M}$ (the tangent bundle of M, restricted to Y, quotiented by the tangent bundle of Y) is diffeomorphic to a small neighborhood of Y.

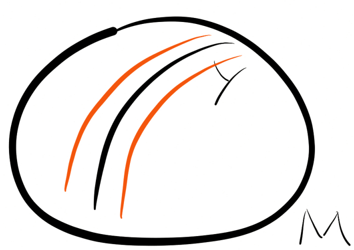

Figure 1: The normal bundle.

We want to think about this picture more dynamically. We will construct a family $F \to \mathbb{C}$ with the following properties. First, notice that there is a $\mathbb{C}^\times$ action here. We want a special point $0 \in \mathbb{C}$ such that the fiber over 0 is $N_{Y/M}$ and such that the fiber over $\mathbb{C}^\times$ is $M \times \mathbb{C}^\times$. So there is a diagram of the form

$$\begin{array}{ccccc} N_{Y/M} & \longrightarrow & F & \longleftarrow & M \times \mathbb{C}^\times \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^\times \end{array} \tag{2}$$

Example Think about circles fibered over $\mathbb{R}$ getting stretched out larger and larger until, at 0, they become two straight lines.

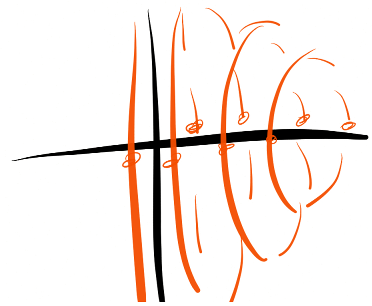

Figure 2: Stretched out circles.

In particular, everything not attached to Y (which is two points here, while the circle is M) is going away to infinity as we get to 0.

More formally, begin with the projection $M \times \mathbb{C} \to \mathbb{C}$. The first step is to blow up $M \times \mathbb{C}$ along $Y \times \{0\}$. The second step is to remove the closure of $(M \setminus Y) \times \{0\}$. The fiber over a nonzero point is M, but the fiber at 0 is the projective space $\mathbb{P}(N_{Y/M} \oplus \text{triv})$ minus the "divisor at infinity," and this is in fact $N_{Y/M}$ again.

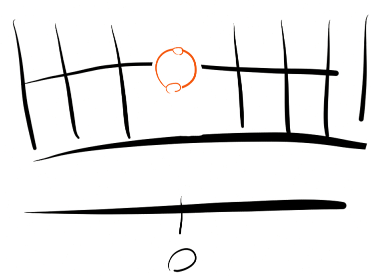

Figure 3: Blowup and removal.

Now we return to the setting of some sheaf F on $M = X \times \mathbb{C}$. We perform the above construction with respect to the submanifold $X \times \{0\}$. This gives us a family over $\mathbb{C}$ which has generic fiber M and special fiber $N_{X/M}$, and we want to send F to $F \boxtimes C^{\bullet}_{\mathbb{C}^\times}$, a sheaf on our family. Finally, we take nearby cycles ψ with respect to this family, giving us a new sheaf F_{con} on $N_{X/M}$. This sheaf is constructible with respect to a $\mathbb{C}^\times$-invariant stratification on the normal bundle, so in particular it is simpler than an arbitrary sheaf.

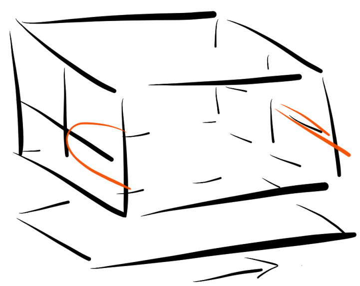

Figure 4: Degeneration to the normal cone.