274 Microlocal Geometry, Lecture 21

David Nadler
Notes by Qiaochu Yuan

Fall 2013

21 More about nearby and vanishing cycles

Consider again the map $f(x,y) = xy$ with singular fiber X_0 and F the constant sheaf. We wrote down a triangle

$$\varphi C_X^\bullet \to C_{X_0}^\bullet \to \psi C_X^\bullet \tag{1}$$

informally, and now we should do it more formally. First, $\psi\mathbb{C}_X^\bullet$ is equipped with a filtration, and we know its associated graded. The first step is $\mathbb{C}_{\text{singpt}}[-1]$. We should think of this as being generated by the cochain surrounding the singular point in a nearby fiber.

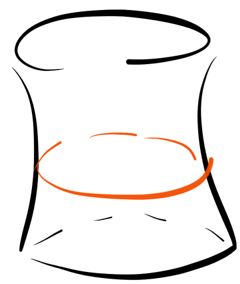

Figure 1: The cochain.

The next step is a sheaf I of cochains which either don't intersect the singular cochain or which contain the entire thing. In particular, the quotient of I by the previous step is IC_{X_0}. The last step in the filtration is $\psi\mathbb{C}_X^\bullet$, and the quotient by I is $\mathbb{C}_{\text{singpt}}[-1]$.

This filtration is compatible with monodromy (it is almost the filtration obtained from the nilpotent part $1 - m_\psi$ of the monodromy). In particular, $I = \ker(1 - m_\psi)$ and $\mathbb{C}_{\text{singpt}}[-1] = \text{im}(1 - m_\psi)$.

We can think about this sheaf using microlocal calculations, in particular computing the microlocal stalk at the singular point. The filtration becomes

$$\mathbb{C}[-1] \xrightarrow{0} \mathbb{C}[-1] \to \mathbb{C}[-1]^{\oplus 2} \tag{2}$$

with the monodromy acting by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{C}[-1]^{\oplus 2}$.

Here is a somewhat more formal definition. If $f : X \to \mathbb{C}$ is a nice map, consider a tube X_B around the singular fiber X_0.

Stratify it with a stratification S. There is a map

$$\pi : X_B \to X_0 \tag{3}$$

which is almost a retraction, and we will define $\psi F = \pi_*(F|_{X_\varepsilon})$, where X_ε is a nearby fiber inside X_B. (Assume that $X \setminus X_0$, as a stratified space, looks like $X_\varepsilon \times \mathbb{C}^\times$.) This

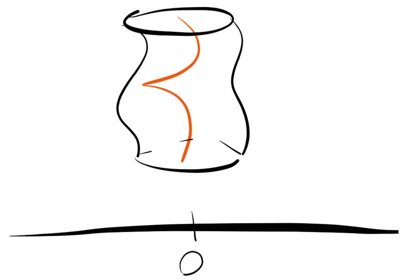

Figure 2: A tube.

construction is done using tube systems. (Almost means that $\pi|_{X_0}$ is not id_{X_0}, but induces the identity on $\mathrm{Sh}_S(X_0)$. This construction is independent of the choice of π.)

Here is an even more formal definition. Nearby cycles does not care about the singular fiber, so we are free to restrict away from it to get a map $X^\times \to \mathbb{C}^\times$. If F is a sheaf on $X^\times$, we can pull back along the universal cover $\exp : \mathbb{C} \to \mathbb{C}^\times$ to get a new sheaf on the pullback $\tilde{X}^\times$. (This is the algebraic geometer's way of taking a nearby fiber without mentioning a specific ε.) We now push this sheaf forward to $X^\times$ again, then to X, then pull it back to X_0. Writing $i : X_0 \to X, j : X^\times \to X$, and $\exp : \tilde{X}^\times \to X^\times$, this gives

$$\psi F = i^* j_* \exp_* \exp^* F. \tag{4}$$

$\exp_* \exp^*$ locally has the effect of replacing fibers by a $\mathbb{Z}$ worth of the same fiber, but globally it unwraps the monodromy.

Here is a standard way to reduce the complexity of the situation. If $f : X \to \mathbb{C}$ is a map, we can factor it into a composite

$$X \xrightarrow{\Gamma_f} X \times \mathbb{C} \xrightarrow{p} \mathbb{C} \tag{5}$$

where the first map is the inclusion of the graph and the second map is a coordinate projection. Starting with a sheaf F on X we can work with the pushforward $(\Gamma_f)_* F$ in relation to the coordinate projection, so we've made the map we want to study simpler at the cost of making the sheaf more complicated. This lets us split up the construction of nearby and vanishing cycles into two steps, the point being that ψ, φ, i_0^* factor through deformation to the normal bundle (we will define this later). In the case of the map $z \mapsto z^2$, the real picture looks like stretching out a parabola to a double line.

Figure 3: Stretching out a parabola.