274 Microlocal Geometry, Lecture 20

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20 Nearby and vanishing cycles

Up to now we've been playing the following game. Given a manifold M with a stratification S, we constructed $T^*_S(M) \subseteq T^*(M)$. Given $(x, \xi) \in T^*_S(M)$ in the smooth locus, we chose nice functions $f_{(x,\xi)} : M \to \mathbb{R}$ and used them to talk about microlocal support and characteristic cycles. Now we're going to play this game with an arbitrary function in place of $f_{(x,\xi)}$. Also, we'll mostly be interested in the holomorphic or complex setting.

Let f be a complex variety and let $f : X \to \mathbb{C}$ be regular. We want to study how the fibers of f change.

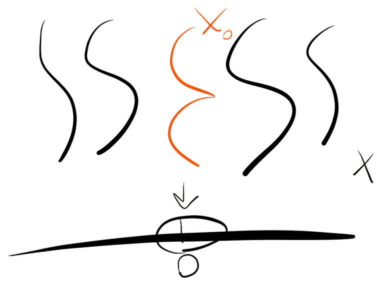

Figure 1: The fibers of a map.

The end result of this study will be a triangle of functors $\mathrm{Sh}(X) \to \mathrm{Sh}(X_0)$. Let $X_0 = f^{-1}(0)$ and let $i_0 : X_0 \to X$ be the inclusion. One functor is restriction $i_0^* : \mathrm{Sh}(X) \to \mathrm{Sh}(X_0)$. Another functor is a functor ψ_f called nearby cycles together with a map $i_0^* \to \psi_f$; roughly speaking this tells us the cohomology of a nearby fiber. Finally, there will be a functor φ_f together with a map $\varphi_f \to i_0^*$ called vanishing cycles. These fit together into a triangle

$$\varphi_f \to i_0^* \to \psi_f \xrightarrow{[1]} . \tag{1}$$

The first is some analogue of (B_0, F_ε), the second is some analogue of B_0, and the third is some analogue of F_ε.

Example Let

$$f : \mathbb{C} \ni z \mapsto z^n \in \mathbb{C} \tag{2}$$

and let $F = C^\bullet_X$. The i_0^* term is $\mathbb{C}^\bullet_{\mathrm{pt}}$ because 0 has one preimage. The nearby term is $\mathbb{C}^\bullet_{F_\varepsilon}$, or the direct sum of n copies of $\mathbb{C}^\bullet_{\mathrm{pt}}$, because a nearby fiber has n points. The vanishing term is the relative cohomology $C^\bullet_{(B,F_\varepsilon)}$, which is (up to some choices) $n-1$ copies of $\mathbb{C}^\bullet_{\mathrm{pt}}[-1]$.

This story depends on the choice of ε: as we choose a nearby point to take the fiber of and vary that point, we get monodromies m_φ and m_ψ of the vanishing and nearby cycles functors.

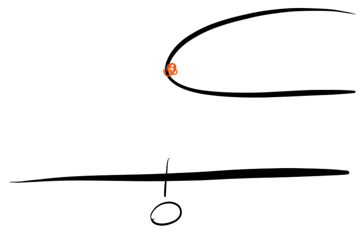

Figure 2: A real picture of $z \mapsto z^2$.

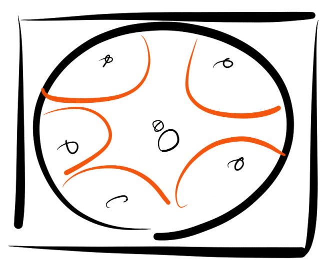

Figure 3: A complex picture of the nearby fiber.

Example Let

$$f : \mathbb{C}^2 \ni (x, y) \mapsto xy \in \mathbb{C} \tag{3}$$

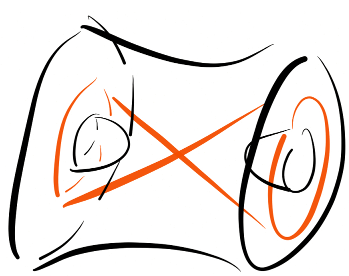

Figure 4: A picture of the fibers of f.

and let $F = C^\bullet_X$. The fiber X_0 over 0 looks like a cone and the nearby fibers look like cylinders. We need to describe three sheaves on X_0. The middle term is the restriction, which is just $C^\bullet_{X_0}$. To describe the others, first let $B_\kappa(x)$ be a small ball around x and let ε be very small. Take

$$\psi_f(F)(B_\kappa(x)\cap X_0) = F(B_k(x)\cap\{f^{-1}(\varepsilon)\}). \tag{4}$$

With $F = C^\bullet_X$, away from the singular point, this sheaf looks like the constant sheaf again. At the singular point we instead assign the cohomology of a cylinder (since that's what a nearby fiber looks like in a ball). Now φ_f is the cone of the map $C^\bullet_{X_0} \to \psi_f C^\bullet_X$. Away from the singular point this vanishes, and at the singular point we have a cochain in degree 2. So this is $\mathbb{C}^\bullet_{\text{singpt}}[-2]$.

(φ_f is dual to some homology that vanishes at X_0, hence the name.)

As ε varies, the monodromy m_ψ vanishes on global sections, hence so does the monodromy m_φ. But m_ψ is not the identity as a sheaf endomorphism; cycles get acted on by a Dehn twist.

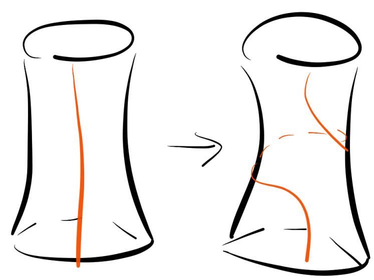

Figure 5: A Dehn twist.

It makes sense to say this because the monodromy can be trivialized at infinity; equivalently, m_ψ is the identity on sheaves away from the singular point. Write $v_\psi = 1 - m_\psi$ for the variation of the monodromy of the identity. The action of v_ψ sends the twisted cycle to a cycle not supported at infinity, and regarded as a map from the stalk at the singular point to the relative stalk, it is nontrivial.

Example Let

$$f : \mathbb{C}^2 \ni (x,y) \mapsto x^2 + y^3 \in \mathbb{C}. \tag{5}$$

The generic fiber is an elliptic curve while the special fiber is a cusp: the extra handle on the elliptic curve degenerates to the cusp.

Here $\psi C^\bullet_X$ is $C^\bullet_{X_0}$ away from the singular point and $C^\bullet$(torus) at the singular point, and $\varphi C^\bullet_X$ is $\mathbb{C}^2_{\text{singpt}}[-2]$.

Question: how do we compute things here?

Answer: one way is to use the Thom-Sebastiani theorem. For $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with 0 the only critical point, the nearby fiber F_ε over a small ε is homotopy equivalent to a wedge

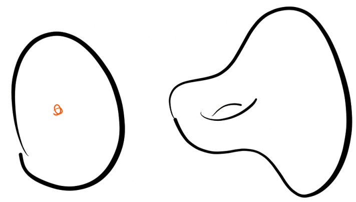

Figure 6: The generic and special fibers.

of spheres. Moreover, if $f(x, y) = g(x) + h(y)$, then F_ε^f is homotopy equivalent to the join $F_\varepsilon^g * F_\varepsilon^h$.