274 Microlocal Geometry, Lecture 2

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2 Whitney stratifications

Yesterday we defined an n-step stratified space. Various exercises could have been but weren't assigned, e.g. that the link of a point has some uniqueness property.

Today we will talk about Whitney stratifications, which in particular provide stratified spaces. Let M be a smooth manifold; it is usually enough to take $M = \mathbb{R}^n$. Let $X \subseteq M$ be a closed subspace of M. Whitney described infinitesimal conditions under which M admits a stratification. First, it needs a decomposition as a disjoint union of strata

$$X = \bigsqcup_{\alpha} S_\alpha \tag{1}$$

which we will assume are connected manifolds. This decomposition should satisfy two conditions: it should be locally finite (every point has a neighborhood which intersects finitely many strata), and the closure of each stratum should be a disjoint union of strata. Finally, the induced decomposition into strata in each open neighborhood should also have the same property.

Non-Example Consider two line segments perpendicular to each other and intersecting at the endpoint of one of the line segments. Then we cannot write down a stratification consisting of one line segment and its complement because the closure of a stratum won't be a union of strata (it will be a union of a stratum and a point).

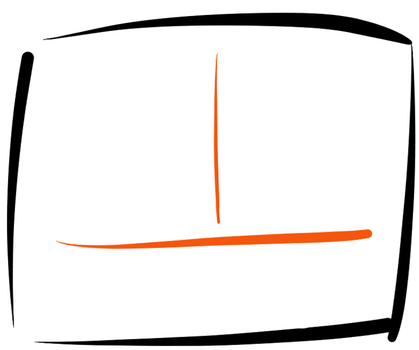

Figure 1: A non-example of the above condition.

Whitney's Condition A can be motivated by looking at the Whitney umbrella again. The unusual point has the property that, as we approach the unusual point, the tangent space of the unusual point is not contained in the tangent spaces of the points by which we approach it (loosely speaking). More precisely, we want the following condition: if S_β, S_α are two strata and $y \in S_\beta, x_i \in S_\alpha$ are points such that $x_i \to y$, then for all subsequences such that the limit $T_{x_i}(S_\alpha)$ exists (in some Grassmannian bundle), it should contain $T_y(S_\beta)$.

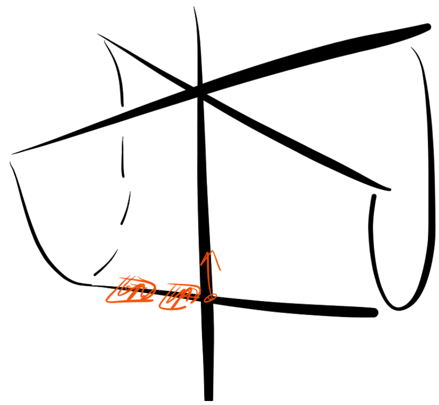

Figure 2: A poorly-behaved sequence of tangent spaces.

Whitney's Condition B can be motivated by looking at the Whitney cusp again. The unusual point has the property that we can find two sequences of points approaching it such that the secant lines between them (we need some extra structure on the ambient manifold to define this) are not contained in the tangent space of the point. More precisely, we want the following condition: if S_β, S_α are two strata, $y_i \in S_\beta, x_i \in S_\alpha$ are two sequences such that the limit of the tangent spaces $T_{x_i}(S_\alpha)$ and the limit of secant lines $\overline{x_i y_i}$ both exist in $T_y(M)$ (again, we need extra structure for this, e.g. a Riemannian metric), then the second limit should be contained in the first limit.

Figure 3: A poorly-behaved sequence of secant lines.

Exercise 2.1. *Condition B does not depend on the choice of extra structure, e.g. a Riemannian metric.*

Definition A *Whitney stratification* of X is a decomposition of X into a disjoint union $\bigsqcup_\alpha S_\alpha$ as above satisfying Whitney's Conditions A and B.

Exercise 2.2. *Condition B implies Condition A.*

The above discussion admits a microlocal interpretation. Recall that thinking microlocally means replacing the study of a manifold M with the study of its cotangent bundle $T^*(M)$. To a submanifold S_α we can associate the conormal bundle $T^*_{S_\alpha}(M)$, which is the bundle of cotangent vectors vanishing on the tangent vectors to S_α. Hence we can associate to X the disjoint union

$$\Lambda = \bigsqcup_\alpha T^*_{S_\alpha}(M). \tag{2}$$

Unfortunately Λ may not even be a closed subset of $T^*(M)$.

Exercise 2.3. *Whitney's Condition A holds iff Λ is closed.*

Please tell me if you know a microlocal interpretation of Condition B!

We want to turn the infinitesimal data above into non-infinitesimal data. Generically the way to do this is ODEs. For example, vector fields have flows for some time ϵ, which give diffeomorphisms.

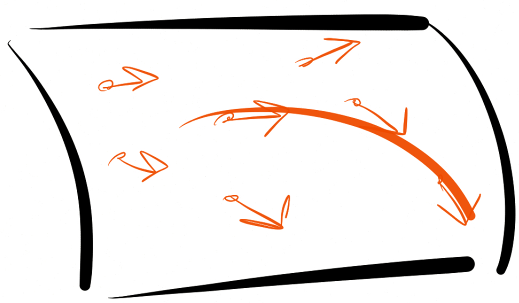

Figure 4: Flowing along a vector field.

To write down diffeomorphisms sending one part of a manifold to another, it suffices to construct appropriate vector fields. For a more concrete example:

Theorem 2.4. *(Ehresmann) Let $f : M \to P$ be a proper (inverse image of compact subspaces is compact) submersion ($df_x : T_x(M) \to T_{f(x)}(P)$ is surjective) of manifolds. Then f is a fibration.*

Proof. (Sketch). Since f is a submersion, we can lift tangent vectors from P to M. Locally this implies that we can lift vector fields, and now we can take the flows of these vector fields. These flows will identify fibers. □

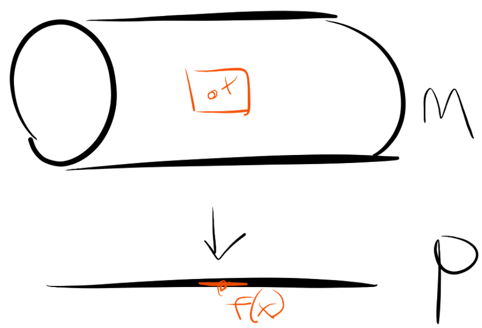

Figure 5: A proper submersion.

Thom generalized this to the case where M is not just a manifold.

Theorem 2.5. *(Thom's first isotopy theorem) Let $f : M \to P$ be a proper smooth map between smooth manifolds. Let X_α be a Whitney stratification of M. Assume that $f|X_\alpha$ is a submersion for all α. Then f is a stratified fibration: each fiber has a stratification, and locally $M \cong P \times F$ (P has a trivial fibration).*

The proof will start out like the previous proof, but we need to make sure vector fields glue nicely, and they might not. Consider the following example involving the Whitney cusp:

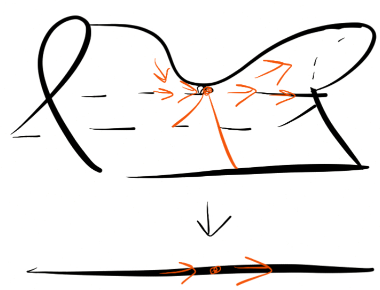

Figure 6: A non-fibration.

This is a proper map which is a submersion with respect to the naive stratification, but the Whitney conditions fail and the map is not a fibration. This is because vector fields do not lift nicely: the lift of the vector field $\frac{d}{dt}$ on the base does not have a flow on the complement of the central line for any positive time ϵ because it cannot intersect the central line.

In the theorem above, the identification $M \cong P \times F$ is a homeomorphism, and this cannot be made any stronger.

Example Consider four planes intersecting at a common line in $\mathbb{R}^3$, projecting down to a line, such that one of the planes is wiggling. After compactifying suitably, this is proper.

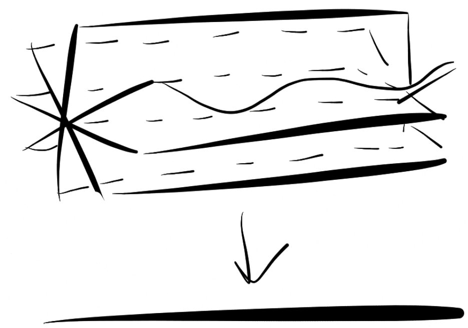

Figure 7: A picture of the space above and its projection to a line.

There is an obvious stratification with respect to which the projection is a stratified submersion, so Thom's theorem asserts that the map is a fibration. The identification $M \cong P \times F$ cannot be made into a diffeomorphism: the fibers cannot be identified smoothly because, with three of the planes fixed, the map on tangent spaces must be a scalar, so cannot correctly deal with the wiggling plane.

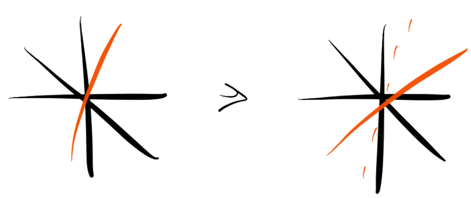

Figure 8: The fibers cannot be smoothly identified.