274 Microlocal Geometry, Lecture 19

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Fall 2013

19 Sheaves on the projective line, the Fourier transform

Last time we described constructible sheaves on $\mathbb{C}$ with a singular point in terms of modules over the quiver

$$\varepsilon^\vee \underset{q[-1]}{\overset{p[1]}{\rightleftarrows}} \varepsilon \tag{1}$$

where $1-qp$ and $1-pq$ are invertible (these give the monodromy up to homotopy, which we can forget since they don't arise in any other conditions). The left vertex is the microlocal stalk $F_{(0,\varepsilon^\vee dx)}$ and the right vertex is the usual stalk $F_\epsilon[-1]$, which we'll shift for convenience so p, q can have degree 0.

Now let's consider sheaves on $M = \mathbb{P}^1$ with a singular point.

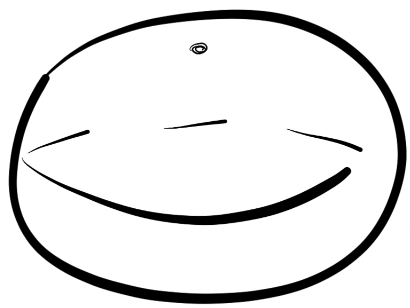

Figure 1: The sphere with a singular point.

This has the effect of adding a 2-cell which kills the monodromy, hence $pq = \delta h$ for some h of degree -1. Letting M_0, M_1 be the stalks above, we want to find indecomposable modules over this quiver with M_0, M_1 in degree 0 ($h = 0$ in this case). There are exactly 5 of these. They can be thought of as perverse sheaves and have something to do with $\mathfrak{sl}_2$-modules of highest weight with central character 0. Here they are:

$$\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0\,,\ \mathbb{C} \underset{0}{\overset{\sim}{\rightleftarrows}} \mathbb{C}\,,\ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}\,,\ \mathbb{C} \underset{\sim}{\overset{0}{\rightleftarrows}} \mathbb{C}\,,\ \mathbb{C}\oplus\mathbb{C} \underset{i_2}{\overset{\pi_1}{\rightleftarrows}} \mathbb{C}\,. \tag{2}$$

Exercise 19.1. *These are all the indecomposable modules.*

The sheaf corresponding to the first module is $\mathbb{C}_{\{0\}}$ and the sheaf corresponding to the third module is $\mathbb{C}_{\mathbb{P}^1}[1]$. The sheaf corresponding to the second module is $j_*\mathbb{C}_U[1]$ where

$j : U \to \mathbb{P}^1$ is the inclusion of $\mathbb{C}$ into $\mathbb{P}^1$. The sheaf corresponding to the fourth module is $j_! \mathbb{C}_U[1]$. Finally, the sheaf corresponding to the fifth module is Harold's pushforward.

Now let's talk about the Fourier transform. Let V be a real vector space and $V^\vee$ its dual. Then $V \times V^\vee$ can be thought of as both $T^*(V)$ and as $T^*(V^\vee)$. In $V \times V^\vee$ consider the subset S of pairs (v, λ) with $\lambda(v) \geq 0$.

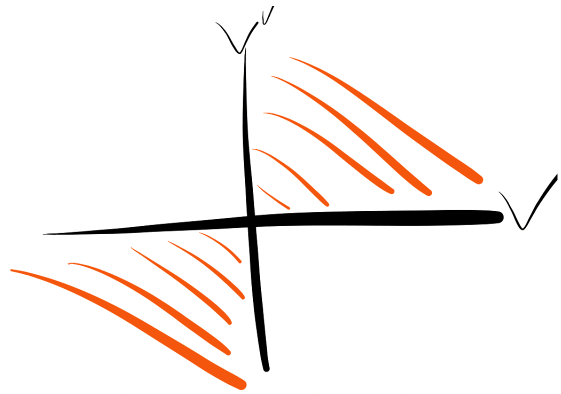

Figure 2: The subset S.

If $C_S^\bullet \in D(V \times V^\vee)$, we can consider the functor

$$D(V) \ni F \mapsto E(F) = (\pi_{V^\vee})_!(C_S^\bullet \otimes \pi_V^* F) \in D(V^\vee). \tag{3}$$

This is an equivalence on sheaves which are $\mathbb{R}_+$-conical. The construction we are performing here should be thought of as analogous to integration against a kernel

$$\hat{f}(\xi) = \int f(x) K(x, \xi)\, dx. \tag{4}$$

Example Let's compute a few examples, all of which may be off by a shift, for $V = \mathbb{R}$. Here $\mathbb{C}_{\{0\}}$ gets sent to $\mathbb{C}_{V^\vee}$. $\mathbb{C}_V$ gets sent to $\mathbb{C}_{\{0\}}$. The pushforward $j_* \mathbb{C}_{(-\infty,0)}$ gets sent to $j_! \mathbb{C}_{(0,\infty)}$.