274 Microlocal Geometry, Lecture 18

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18 Sheaves on the affine line

Let $M = \mathbb{C}$ with stratification the origin and its complement. We want to study $\mathrm{Sh}_S(M)$. Recall that a constructible sheaf is described by giving its stalk F_0 at 0, its stalk F_1 at 1, the restriction map $r : F_0 \to F_1$, and the monodromy map $m : F_1 \to F_1$. Moreover, mr is homotopic to r via a homotopy given by a map $h : F_1 \to F_1$ of degree -1. This description is somewhat unsatisfactory in that it does not see some symmetries of the situation. We will give a better microlocal description.

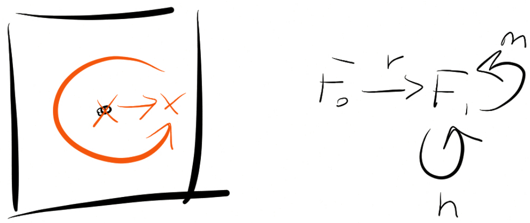

Figure 1: Previous description of sheaves on the affine line.

The cotangent bundle $T^*(M)$ looks like $\mathbb{C}^2$. There are two coordinate axes which are in some sense on equal footing. Microlocally, we will still measure the stalk F_ϵ at a nonzero point, but we also want to measure the microlocal stalk $F_{\epsilon^\vee}$ at some smooth cotangent vector $(0, \epsilon^\vee dx)$.

Example Let $F = C^\bullet_M$. Then $F_{\epsilon^\vee} = 0$ but $F_\epsilon = \mathbb{C}$.

Example Let $F = \mathbb{C}_{\{0\}}$. Then $F_{\epsilon^\vee} = \mathbb{C}$ but $F_\epsilon = 0$.

Microlocally these look very similar, whereas in the usual picture they might not. Moreover, $F_\epsilon, F_{\epsilon^\vee}$ together are faithful in the sense that if they both vanish then $F = 0$.

In addition to these two stalks there are also two monodromies $m : F_\epsilon \to F_\epsilon$ and $m^\vee : F_{\epsilon^\vee} \to F_{\epsilon^\vee}$.

Example Let $j : \mathbb{C}^\times \to \mathbb{C}$ be the inclusion and let $F = j_* L_\alpha$ where L_α is the local system with monodromy given by multiplication by some $\alpha \in \mathbb{C}^\times$ not equal to 1. Then $F_0 = 0$ because the nontrivial monodromy means there are no sections on a small ball. $F_\epsilon = \mathbb{C}$, and $F_{\epsilon^\vee} = \mathbb{C}[-1]$ because we are computing the relative cohomology of a circle relative to a point. The monodromy m is multiplication by α, and so is the monodromy $m^\vee$.

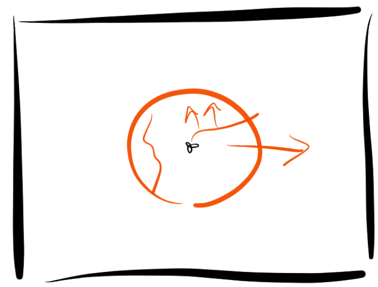

Figure 2: A microlocal stalk.

Note that there is a natural triangle

$$F_{\epsilon^\vee} \to F_0 \to F_{-\epsilon}. \tag{1}$$

Example Here is an example where m and $m^\vee$ disagree. Let F be the sheaf which is constant away from the origin and which, at the origin, is cochains which stay away from half of (small balls around) the origin. Then $F_\epsilon = \mathbb{C}$ with identity monodromy because it's away from the origin. $F_0 = \mathbb{C}[-1]$ because it is again the relative cohomology of a circle relative to a point. And $F_{\epsilon^\vee} = \mathbb{C}^2[-1]$ because it is the relative cohomology of a circle relative to two points.

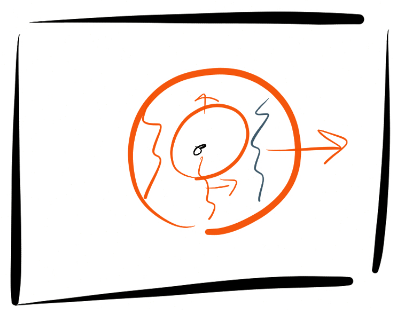

Figure 3: An interesting sheaf.

The monodromy $m^\vee$ is interesting. With an appropriate choice of basis, it turns out to be $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; in particular, it is not the identity.

The natural triangle tells us that there is a coboundary map $\delta : F_\epsilon \to F_{\epsilon^\vee}$ of degree 1. There is also a map $\sigma : F_{\epsilon^\vee} \to F_\epsilon$. In homology it looks like the following. With B a disk and N half a disk, there is a boundary map from relative chains $C_\bullet(B, N)$ (which like N) to chains $C_\bullet(N)$ of degree -1. There is also a map in the other direction of degree 1 which, given a homology class on N, sweeps it out along a path through B back to N to get a relative homology class on B of one higher degree.

The dual maps in cohomology look like the following. The coboundary map looks like a kind of thickening or Gysin map. The dual to the sweeping map is a relative restriction.

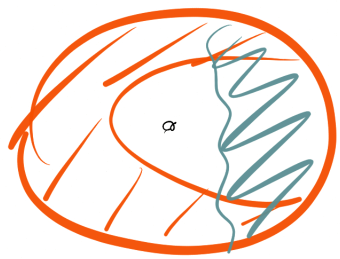

Figure 4: The cosweeping map.

Example Consider again $F = j_* L_\alpha$. Whether or not $\alpha = 1$ we always have $F_\epsilon = \mathbb{C}$ and $F_{\epsilon^\vee} = \mathbb{C}[-1]$, and moreover both monodromies m are equal to α.

Now we want to calculate δ and σ. When $\alpha = 1$, the coboundary map δ is zero (a thickened cochain can be sent away), but the cosweeping map σ is an isomorphism. When $\alpha \neq 1$, δ is now an isomorphism because the nontrivial monodromy prevents us from sending a thickened cochain away, and so is σ. With a suitable choice of generators, $\delta = 1 - \alpha$ and $\sigma = 1$.

We can now ask about relations between the maps we wrote down. Let's rename δ to p and rename σ to q. Then it turns out that $1 - pq = m^\vee$, but homotopically: there is an $h^\vee$ such that $\delta h^\vee = (1 - pq) - m^\vee$. Similarly, $1 - qp = m$, but homotopically: there is an h such that $\delta h = (1 - qp) - m$. This turns out to be all of the data in a constructible sheaf on M.

Note that the description we've given of a constructible sheaf on M is now completely symmetric: we can exchange the roles of F_ϵ and $F_{\epsilon^\vee}$. There is a kind of Fourier transform here that would be hard to see in the usual description of constructible sheaves on M.

Exercise 18.1. *Assume $F_{\epsilon^\vee}$ is concentrated in degree 0 and F_ϵ is concentrated in degree -1. Classify the possible sheaves.*

Exercise 18.2. *Assume in addition that m is the identity. Show that there exist five indecomposable sheaves.*

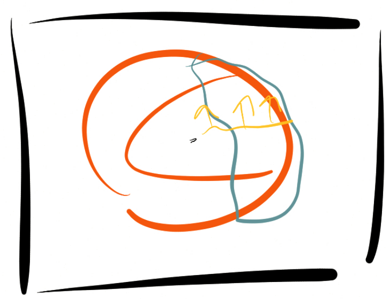

Figure 5: The cosweeping map.