274 Microlocal Geometry, Lecture 17

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Fall 2013

17 Computations with characteristic cycles

Last time we discussed isomorphisms

$$K(\mathrm{Sh}_S(M)) \cong \mathrm{Fun}_S(M) \cong H^{BM}_{\mathrm{top}}(T^*_S(M)). \tag{1}$$

The first isomorphism was given by taking local Euler characteristics χ, while the second was given by taking microlocal Euler characteristics $\mu\chi$ (and twisting by orientation). In particular, $\mu\chi(F)$ only depends on $\chi(F)$. We can see this by constructing an inverse. Let $x \in M, B_x \ni x$, and $j : B_x \to M$ be the inclusion. Then $\Gamma(B_x, F) \cong \mathrm{Hom}(j_!\mathbb{C}_{B_x}, F)$. We can compute its Euler characteristic by intersecting the characteristic cycle $\mu\chi$ of $j_!\mathbb{C}_{B_x}$ (possibly after applying some dualization) and the characteristic cycle of F.

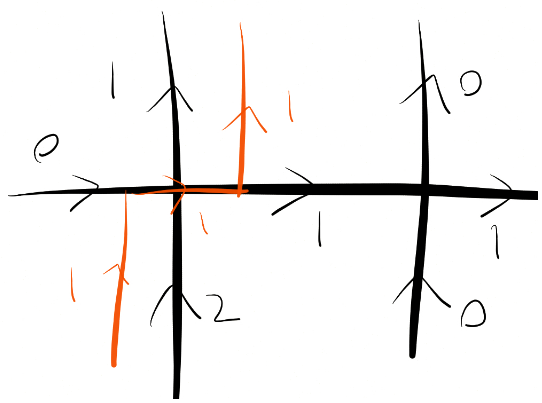

Figure 1: Intersecting cycles.

One curious fact about this story is that constructible functions have an obvious basis given by the indicator functions on each stratum. Characteristic cycles don't have such an obvious basis. However, characteristic cycles are better in that they make some symmetries more visible.

Now let's do some calculations.

Example Let $M = \mathbb{C}^2$ and consider the complex cusp $C = \{y^2 = x^3\}$ in it. Let F be intersection homology. C is homeomorphic to its normalization, which is just $\mathbb{A}^1$. However, it is embedded in an interesting way. The intersection of a small sphere around the cusp with C turns out to be the trefoil knot. In particular, around the cusp it is far from being a submanifold.

The only interesting thing to calculate is at the cusp. At the cusp there is a (complex) 2-dimensional conormal space. The smooth locus here is the complement of a (complex) 1-dimensional subspace, and in particular it is connected. Hence our computation will not depend on the choice of covector. However, there will potentially be interesting monodromy as we move our covector around. If $B_x \ni x$ is a small ball, the intersection of B_x with the

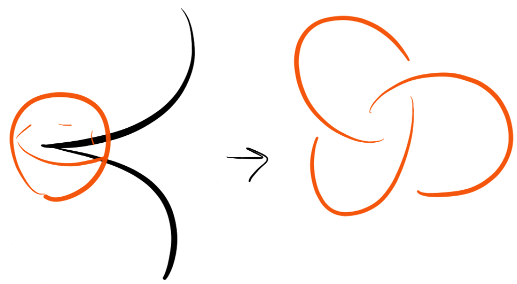

Figure 2: A real picture of the complex cusp and the trefoil knot.

curve will be a disc, and the intersection of N_x with the curve will be two arcs. Hence there is one interesting cochain in degree 1, and $F_{(x,\xi)} \cong \mathbb{C}[-1]$.

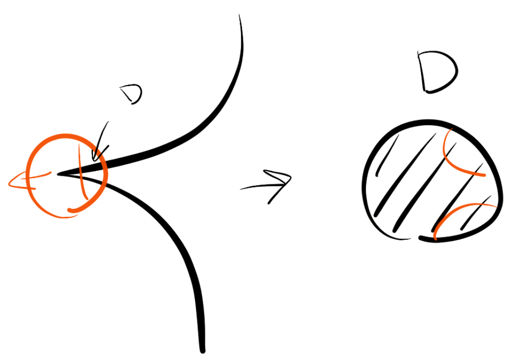

Figure 3: Microlocal calculations at the cusp.

What happens as we move ξ around? The intersection of N_x with the curve twists around, and this induces a monodromy of -1.

Example Let $M = \mathbb{C}^3$ and consider the quadratic cone $Q^2 = \{xz = y^2\}$. The intersection of Q^2 with a small sphere S^5 around the cone point is $\mathbb{RP}^3$ based on looking at the action of SO(3). In particular $\mathbb{RP}^3$ is close to S^3, so Q^2 is close to smooth - it is rationally smooth.

We'll compute microlocal Euler characteristic for two sheaves which are equal in the Grothendieck group but which have different microlocal stalks. First, let j be the inclusion $Q^2_{\text{sm}} \to Q^2$ of smooth points, and let $F = j_* C^\bullet_{Q^2_{\text{sm}}}$. The only interesting thing happens at the cone point. Here the smooth conormal vectors are $\mathbb{C}^3$ minus a dual light cone. The

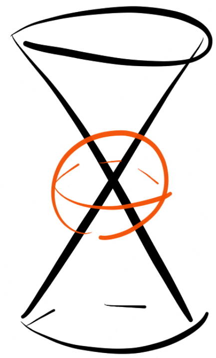

Figure 4: A real picture of the quadratic cone.

sheaf F is not supported at the cone point, so we can push things onto the link $\mathbb{RP}^3$ to do calculations.

Let (x, ξ) be a smooth point, where x is the cone point. $B_x \cap Q^2_{\mathrm{sm}}$ should be $\mathbb{RP}^3 \times (0, 1)$. What is $N_{(x,\xi)}$? It should be a complex hyperbola. After being pushed into $\mathbb{RP}^3$ it should be a copy of $\mathbb{RP}^1 \cong S^1$. So we want to compute the cohomology of $\mathbb{RP}^3$ relative to $\mathbb{RP}^1$. Over $\mathbb{C}$ this is $\mathbb{C}$ in dimensions 2 and 3, and it vanishes otherwise. In particular, the Euler characteristic is zero, so the characteristic cycle is not supported at (x, ξ) even though (x, ξ) is part of the singular support. The smooth conormal vectors have some interesting π_1 so there is some interesting monodromy which we will not calculate.

Now, Q^2_{sm} is homotopy equivalent to $\mathbb{RP}^3$, so its π_1 is $\mathbb{Z}_2$. Hence we can consider a nontrivial local system L on Q^2_{sm} and repeat the above story. This gives $F_{(x,\xi)}$ as the relative cohomology of $\mathbb{RP}^3$ with respect to $\mathbb{RP}^1$ with coefficients in L. The cohomology of both $\mathbb{RP}^1$ and $\mathbb{RP}^3$ with coefficients in L vanishes, hence so does this relative cohomology. In particular (x, ξ) is not part of the singular support.