274 Microlocal Geometry, Lecture 16

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16 Characteristic cycles

Let M be an oriented manifold with stratification S, let F be an S-constructible sheaf on M, and let $(x, \xi) \in T^*_S(M)$ be a smooth point. What can we say if we are only interested in the Euler characteristic $\chi(F_{x,\xi})$?

Recall that we claimed there is an isomorphism from the Grothendieck group $K(\mathrm{Sh}(M))$ to constructible functions on M given by sending a sheaf F to the constructible function which assigns to $x \in M$ the Euler characteristic of the stalk F_x.

Question: consider the fibration $f : Kl \to S^1$ of the Klein bottle over S^1 and consider the sheaf $f_* C^\bullet_{Kl}$. It looks like the constructible function $\chi(f_* C^\bullet_{Kl})$ is zero, but is this sheaf really zero in the Grothendieck group?

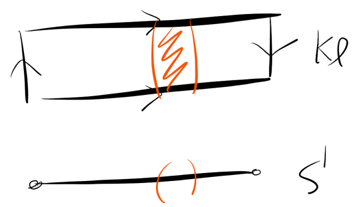

Figure 1: The Klein bottle.

Answer: yes, it is really zero in the Grothendieck group. To see this we can first write it as a direct sum $C^\bullet_{S^1} \oplus C^\bullet_{S^1}(\text{fiber or})[-1]$. We can cut up these sheaves by considering a point $\mathrm{pt} \in S^1$ with complement U and looking at the distinguished triangles associated to the inclusions $U \xrightarrow{j} S^1 \xleftarrow{i} \mathrm{pt}$ which settles the issue (and this is how the theorem is proven in general).

We would like a microlocal version of this story telling us something about characteristic cycles $\chi(F_{x,\xi})$. First, observe that $F \mapsto F_{(x,\xi)}$ and $F_{(x,\xi)} \to \chi(F_{x,\xi})$ both behave nicely with respect to distinguished triangles. Hence $\chi(F_{x,\xi})$ factors through the Grothendieck group, so equivalently it must factor through constructible functions.

Example Consider the real cusp in $\mathbb{R}^2$ and let F be the constant sheaf on the curve. The constructible function assigns 1 to every point on the curve and 0 otherwise. Now let x be the cusp and let ξ be a non-vertical cotangent vector at x. Then $F_{x,\xi} = \mathbb{C}[-1]$ twisted by coorientation, and $\chi(F_{x,\xi}) = -1$. The claim is that we can get this number from the constructible function above.

We can do this as follows. $\chi(F_{x,\xi})$ is the relative Euler characteristic of F on a small ball B_x with respect to the region $N_{(x,\xi)} = \{f = -\epsilon\}$ for small ϵ. So we can compute it by subtracting these two. On B_x this is the integral of the constructible function we get with respect to the Euler characteristic (we split up into parts and add up each part weighted by

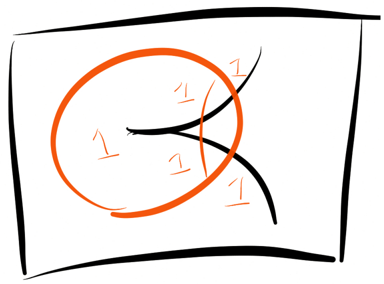

Figure 2: The real cusp in the plane.

the Euler characteristic). We get 3 points and 2 line segments, so the answer is $3 - 2 = 1$. On $N_{(x,\xi)}$ we get 2 points, so the answer is 2. Overall the answer is $1 - 2 = -1$.

This gives us a map which assigns, to any sheaf, a function $(x, \xi) \mapsto \chi(F_{x,\xi})$ (the characteristic cycle $\mu\chi$ of F) on the smooth locus of $T^*_S(M)$. This map factors through the Grothendieck group, and we claim that as a map on the Grothendieck group it is injective. This is essentially the content of the following lemma.

Lemma 16.1. *Let f be a nonzero constructible function and let S_α be a stratum which is open in the support of f. Then for any $x \in S_\alpha$ and $(x, \xi) \in T^*_{S_\alpha}(M)$ we will have*

$$(f)_{x,\xi} = \int_{B_x} f - \int_{N_{(x,\xi)}} f \neq 0. \tag{1}$$

(Since we observed that $F \mapsto \chi(F_{x,\xi})$ factors through constructible functions we can regard it as acting on constructible functions.)

What is the image of the characteristic cycle map $\mu\chi$? It turns out to be conical Lagrangian cycles. This is the union of top Borel-Moore cycles $Z^{BM}_{\text{top}}(T^*_S(M))$ over all stratifications S.

One way to interpret the computation of $\chi(F)$ (the ordinary Euler characteristic of a sheaf F) that we did last time is that it is the intersection $\Gamma_{df} \cap \mu\chi(F)$ of cycles.