274 Microlocal Geometry, Lecture 15

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15 Microlocal Morse theory

Let M be a manifold with a stratification S and F an S-constructible sheaf on M. Last time we associated, to every point $(x, \xi) \in T^*_S(M)$, a complex $F_{(x,\xi)}$ using a function $f : B_x \to \mathbb{R}$, where $B_x \ni x$ is a small ball.

When $S = \{M\}$ this is more or less Morse theory. In this setting F is a local system. Suppose M is compact. We might want to calculate the (derived) global sections of F. We can do this using open covers but this is messy. Instead we will choose a Morse function $f : M \to \mathbb{R}$ and look at its critical points, which are the intersection $\Gamma_{df} \cap M$ in $T^*(M)$. A Morse function equips $\Gamma(M, F)$ with a filtration by $\Gamma(M_{f \le c}, F)$, and

1. if $c \le c'$ and there are no critical points between them, then the comparison map from $\Gamma(M_{f \le c'}, F) \to \Gamma(M_{f \le c}, F)$ is a quasi-isomorphism as long as there are no critical points between c and c', and
2. if there is a critical point $c \le f(p) \le c'$ between them, then relative sections $\Gamma((M_{\le c'}, M_{\le c}), F)$ is the stalk F_p of F at the critical point together with a shift and a twist.

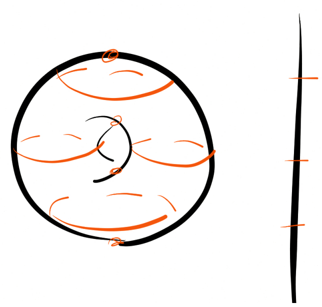

Figure 1: An example of a Morse decomposition.

Example Consider the sphere S^2 equipped with a Morse function giving it two horns. Going up a few critical points gives us a long exact sequence relating the cohomology of a ball, the cohomology of a circle, and the cohomology of a circle relative to a ball. We can calculate the first and third thing and this lets us calculate the second thing.

We haven't really calculated $\Gamma(M, F)$ because we would need to know some boundary maps, but at least we can calculate the Euler characteristic as a sum over critical points.

We would like a microlocal analogue of this story when M comes with a stratification S. Ordinarily $f : M \to \mathbb{R}$ being a Morse function is equivalent to Γ_{df} being transverse to the zero section in $T^*(M)$. With the stratification, let us now say that f is Morse if Γ_{df} is transverse to $T^*_S(M)$; their intersection is our new version of critical points. If F is an S-constructible sheaf, $\Gamma(M, F)$ again admits a filtration by $\Gamma(M_{\le c}, F)$ and we would like the analogues of the standard statements in Morse theory, namely that

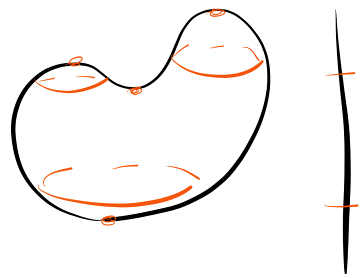

Figure 2: A Morse decomposition of the sphere.

1. if $c \leq c'$ and there is no critical point between them then $\Gamma(M_{\leq c'}, F) \to \Gamma(M_{\leq c}, F)$ is a quasi-isomorphism, and
2. if $c \leq f(p) \leq c'$ with one critical point (x, ξ) in between then $\Gamma((M_{\leq c'}, M_{\leq c}), F)$, suitably defined, should be $F_{(x,\xi)}$.

Once we know this, we get an index theorem, the Dobson-Kashiwara index theorem, giving $\chi(M, F)$ as an alternating sum of Euler characteristics $\chi(F_{(x,\xi)})$. All of this is proven using the usual Morse theory proofs via the Thom isotopy lemma.

Example Consider the kissing banana with a vertical Morse function.

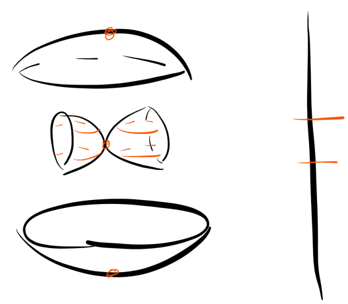

Figure 3: X-ray vision of a Morse decomposition of the kissing banana.

We will think of it as a sphere S^2 with two points squashed together on the inside. This Morse function has three critical points. Let M be intersection cohomology (recall that this is the same as the cohomology of S^2). At the bottom, $M_{(x,\xi)} \cong \mathbb{C}$ in degree zero. At the top, $M_{(x,\xi)} \cong \mathbb{C}$ in degree two with a twist. In the middle, $M_{(x,\xi)} \cong 0$; intersection cohomology is not sensitive to this critical point.

Now let F be the constant sheaf. At the bottom and the top it's the same as above. In the middle there is a 1-cochain stuck around the singular point, so $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1.