274 Microlocal Geometry, Lecture 14

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14 More about microlocal support

Let $Y \subseteq X$ be a submanifold, 0 a point on Y, and ξ a covector in the conormal bundle. Write y for (multi-)coordinates in the Y direction and x for (multi-)-coordinates normal to Y. Last time we considered functions $f : X \to \mathbb{R}$ such that $f(0) = 0, df_0 = \xi$, and such that the graph Γ_{df} of df is transverse to $T^*_Y(X)$. The first condition means that the Taylor series expansion of f has no constant term. The second condition means the linear term is ξx. The third term means that the quadratic term is nondegenerate in y, e.g. $f(x, y) = \xi x + Ay^2 + ...$ (where A is a matrix, which we may later want to require to be positive-definite), so f behaves like a Morse function along y.

Let $F \in \mathrm{Sh}_S(X)$. To each (x, ξ) in the smooth locus of $T^*_S(X)$ we will assign a complex $F_{(x,\xi)}$ as follows. If B_x is a small ball around x, we choose a function $f_{(x,\xi)} : B_x \to \mathbb{R}$ satisfying the above conditions such that along the stratum containing x it has a local minimum at f. Letting $N_{(x,\xi)} = \{f = -\epsilon\} \subset B_x$, we want to consider relative sections $F(B_x, N_{(x,\xi)})$. This fits into a long exact sequence of the form

$$F(N_{(x,\xi)}) \leftarrow F(B_x) \leftarrow F_{(x,\xi)}. \tag{1}$$

Exercise 14.1. *$F_{(x,\xi)}$ is independent of f up to quasi-isomorphism.*

We can always ask whether $F_{(x,\xi)}$ is quasi-isomorphic to zero or what its Euler characteristic is. The former gives us singular support $\mathrm{ss}(F)$ and the latter is the value of the characteristic cycle $\mathrm{cc}(F)$.

Example Let $S = X = \mathbb{R}^2$ and let $F = C^\bullet_X$ be cochains. If $\xi \neq 0$ then the restriction map induced by the inclusion $N_{(x,\xi)} \to B_x$ is always a quasi-isomorphism independent of the choice of f, so nothing depends on f. If $\xi = 0$ then we have a choice of three possible quadratic terms $f = x_1^2 + x_2^2, f = x_1^2 - x_2^2, f = -x_1^2 - x_2^2$. In the first case $N_{(x,\xi)}$ is empty and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree zero. in the second case $N_{(x,\xi)}$ has two components and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1, twisted by coorientation (so the Euler characteristic is different). In the third case $N_{(x,\xi)}$ is an annulus and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 2, again twisted by coorientation.

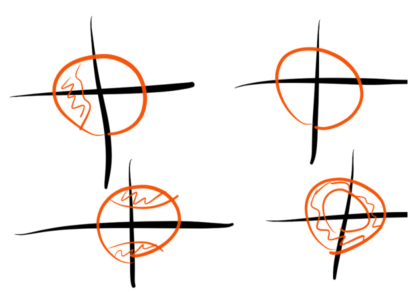

Figure 1: Dependence on the quadratic part.

Example Let $X = \mathbb{R}^2$ stratified by a cuspidal curve γ. Let $F = C^\bullet_\gamma$ be the constant sheaf on the curve. This sheaf is constructible. With respect to this stratification, $T^*_S(X)$ is the zero section on the smooth locus, consists of normal covectors on the smooth locus of γ, and is the whole cotangent space on the cusp 0. The smooth covectors are the covectors on the smooth locus, the covectors (y, ξ) on the smooth locus of γ with $\xi \neq 0$, and the covectors $(0, \xi)$ where ξ is not a limit of covectors on the smooth locus. (This happens generally.)

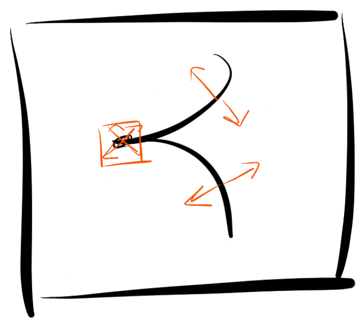

Figure 2: Cotangent vectors.

The computation on the smooth locus is uninteresting. On the smooth locus of γ, let t, s be local coordinates where t points along the curve. We want to pick an f of the form $\xi s + t^2$. Then $N_{(x,\xi)}$ doesn't intersect the curve, so $F_{(x,\xi)}$ is $\mathbb{C}$ concentrated in degree zero.

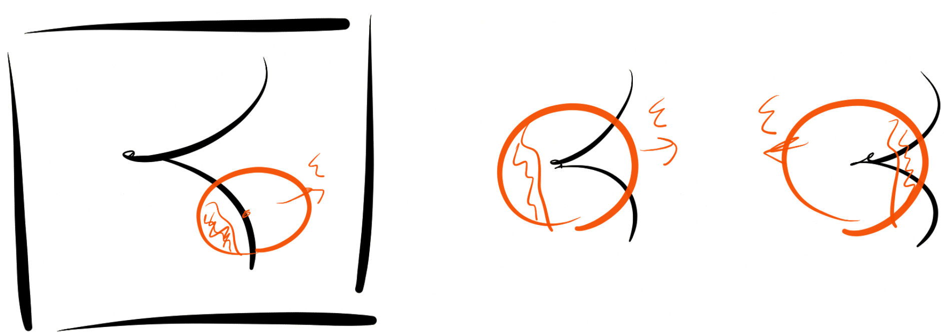

Figure 3: On the smooth locus of γ and at the cusp.

At the cusp two things can happen. Either $N_{(x,\xi)}$ does not intersect the curve, so again $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 0, or $N_{(x,\xi)}$ intersects the curve in two components, so $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1, again twisted by coorientation.

Altogether the singular support consists of all covectors in $T^*_S(X)$ on γ.

Example Let X be as above, but consider the sheaf F obtained from the constant sheaf on half of the complement of the curve, pushed forward to the bottom half of the curve by $*$ and pushed forward to the top half by !. These are cochains which can touch the bottom half of the curve but not the top half. What happens on the smooth locus and the smooth locus of the curve is similar to the above, except that the choice of ! rather than $*$ pushforward switches the behavior.

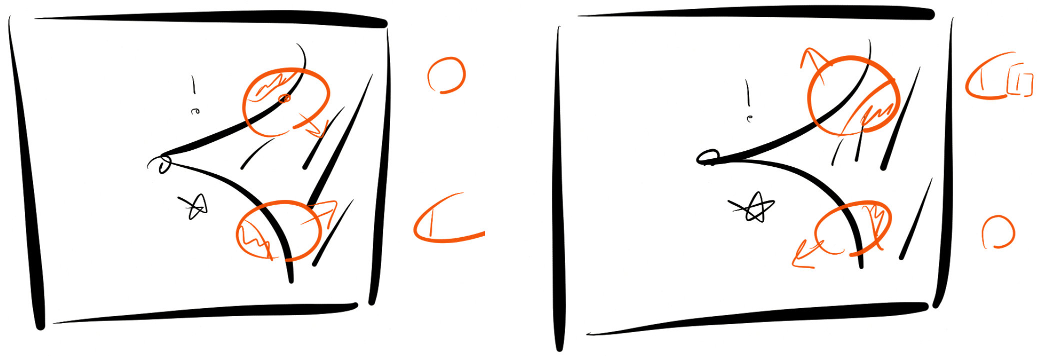

Figure 4: On the smooth locus of the curve.

Something interesting happens at the cusp. The singular support contains the conormal vector pointing upward by closure. But in fact it contains no other conormal vectors; in all other cases the conditions on boundary behavior make $F_{(x,\xi)}$ quasi-isomorphic to zero.

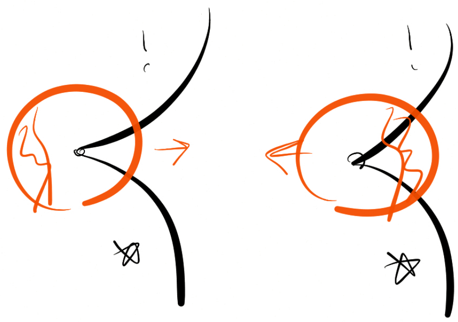

Figure 5: At the cusp.