274 Microlocal Geometry, Lecture 12

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12 More about constructible sheaves

Let X be a reasonable space, $i : Y \to X$ be the inclusion of a closed subspace, and $j : U \to X$ be the inclusion of its open complement. Last time we wrote down a distinguished triangle $i_! i^! F \to F \to j_* j^* F \xrightarrow{[1]}$. The operations $j_* j^*$ ere naive operations but $i^!$ was a somewhat more sophisticated operation. With $F = C_X^\bullet$ and $Y \subseteq X$ a submanifold of a manifold X this took the form

$$C_Y^\bullet \otimes \mathrm{or}_{Y/X}[-\mathrm{codim}_{Y/X}] \to C_X^\bullet \to C_U^\bullet \xrightarrow{[1]} . \quad (1)$$

Taking cohomology, the first term can be reinterpreted as the relative cohomology $H^\bullet(X; U)$, and this gives us a long exact sequence in cohomology.

The importance of this sequence is that it tells us we can build constructible sheaves on a stratified space inductively in terms of pushforwards of local systems on the strata. In particular we will be able to give Deligne's definition of intersection cohomology in the language of sheaf theory.

Let X be a space stratified by even-codimensional strata. We will be indexing strata by codimension. Let $j_0 : S^0 \to X$ be the inclusion of the 0-codimensional stratum. Let $F^0 \in \mathrm{Loc}(S^0)$ be a local system concentrated in degree 0. Consider the (derived) pushforward $(j_0)_* L^0 \in \mathrm{Sh}(X)$. This computes the cohomology of a small ball intersect the open stratum (with coefficients in F^0). Consider the restriction $\tilde{F}^{\leq 2}$ of this pushforward to $S^0 \cup S^2$. We will cohomologically truncate this sheaf; this is the throwing out of certains chains and cochains that we did earlier. More precisely we take $F^{\leq 2} = \tau^{\leq 0} \tilde{F}^{\leq 2}$ where $\tau^{\leq 0}$ replaces a complex

$$\cdots \to A^{-1} \to A^0 \xrightarrow{d^0} A^1 \to \cdots \quad (2)$$

with

$$\cdots \to A^{-1} \to \ker(d^0) \to 0 \to \cdots . \quad (3)$$

(This is adjoint to the inclusion of complexes which are zero in positive degree into all complexes.) We now repeat this construction: we'll push $F^{\leq 2}$ forward along the inclusion $j_{\leq 2} : S^0 \cup S^2 \to X$ to get $\tilde{F}^{\leq 4}$, then take a truncation $F^{\leq 4} = \tau^{\leq 1} \tilde{F}^{\leq 4}$ in degrees less than or equal to 1, etc. Then intersection cohomology with coefficients in F^0 is the final output $F^{\leq \dim X}$ of this construction. In particular it is a constructible sheaf. Geometrically $F^{\leq 4}$ corresponds to only allowing codimension 0 and 1 cochains to intersect S^4.

If we try to apply the distinguished triangle we did earlier to intersection cohomology, $j^* F$ is intersection cohomology again, but $i^! F$ contains complicated information. It is an interesting question to try to write intersection cohomology in terms of local systems on the strata.

There is a distinguished triangle dual to the one we wrote above. It takes the form

$$j_! j^! F \to F \to i_* i^* F \xrightarrow{[1]} \quad (4)$$

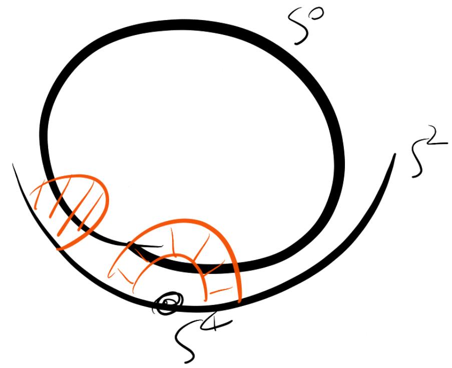

Figure 1: Sections.

where now we've switched the roles of i and j. Here $i_* = i_!$ because i is the inclusion of a closed subspace and $j^! = i^*$ because j is the inclusion of an open subspace. $j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ is universal with the property that sections on any small ball around a point in Y are zero.

Example Let $Y \subseteq X$ be the inclusion of a submanifold into a manifold and let $F = C_X^\bullet$ again. Then the distinguished triangle takes the form

$$C_{X;Y}^\bullet \to C_X^\bullet \to C_Y^\bullet \xrightarrow{[1]} \tag{5}$$

where $C_{X;Y}^\bullet$ is cochains away from Y, and passing to cohomology gives another long exact sequence in cohomology.

So far we've encountered four of Grothendieck's six operations. Let $f : X \to Y$ be a reasonable map between reasonable spaces. This induces two (derived) adjunctions (f^*, f_*) and $(f_!, f^!)$ between categories of constructible sheaves. $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is the (derived) pullback. $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the (derived) pushforward. $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the (derived) pushforward with proper support: here, if $U \subseteq Y$, then $f_! G(U)$ would be sections of $G(f^{-1}(U))$ such that f restricted to their support is a proper map.

$f^!$ is the right adjoint to (derived) pushforward with proper support. For a closed inclusion it's the same as f^*, and we also described it for an open inclusion. We will also describe $f^!$ when $f : X \to \mathrm{pt}$; in general $f^!$ is defined fiberwise. $f^!$ of the constant sheaf on the point is shifted Borel-Moore chains (closed support, not compact support) $C_{-\bullet}^{BM}[\dim X]$. It is sometimes called the Verdier dualizing complex on X.

The claim that this is the right adjoint of $f_!$ implies in particular that

$$\mathrm{Hom}(f_! C_X^\bullet, \mathbb{C}_{\mathrm{pt}}) \cong \mathrm{Hom}(C_X^\bullet, f^! \mathbb{C}_{\mathrm{pt}}). \tag{6}$$

Morphisms from a constant sheaf give global sections, so the RHS is just Borel-Moore homology $H_{-\bullet}^{BM}(X)$. The LHS is the dual of cohomology with compact support $H_c^\bullet(X)^\vee$.

This is a version of the standard duality between homology and cohomology. (This has nothing to do with Poincaré duality. To get Poincaré duality we need to know that on a compact manifold we have $f^! \cong f^* \otimes \mathrm{or}[\dim X]$.)

(The other two of Grothendieck's operations are sheaf tensor and sheaf hom.)

Some identities between the operations that are used in practice:

1. If f is proper, then $f_! = f_*$.
2. If f is a fibration with smooth fibers, then $f^! = f^* \otimes \mathrm{or}[\dim F]$ where F is the fiber.
3. In particular, if f is an open inclusion, then $f^* = f^!$.

What also gets used in practice is base change. Consider a pullback square

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{s'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{s} & Y \end{array} \quad . \tag{7}$$

If F is a sheaf on X, we can try to turn it into a sheaf on Y' by passing through Y or by passing through $X \times_Y Y'$. These two possibilities are compatible in the sense that $s^! f_* F \cong (f')_* (s')^! F$ and $s^* f_! F \cong (f')_! (s')^* F$. This is not true if all of the shrieks are replaced with stars.

Example Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $f : X \to \mathbb{R}$ be the projection onto the x-axis. Let $s : \mathrm{pt} \to \mathbb{R}$ be the inclusion of the origin, so the pullback is the fiber X' over the origin of f.

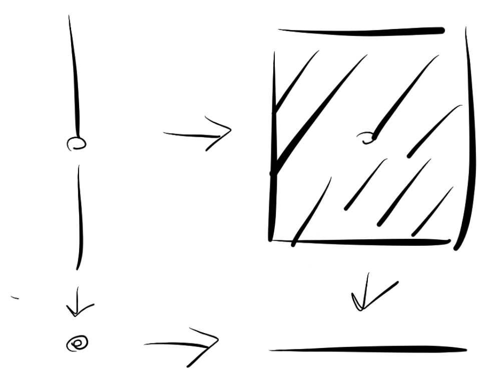

Figure 2: The spaces above.

Let $F = C^\bullet_X$. Then

1. $f_* C^\bullet_X$ is $C^\bullet_{\mathbb{R}} \oplus C^\bullet_{\{0\}}[-1]$,

2. $(s')^* C^\bullet_X$ is $C^\bullet_{X'}$,
3. $(f')_* C^\bullet_X = \mathbb{C} \oplus \mathbb{C}$, but
4. $s^* \left(C^\bullet_{\mathbb{R}} \oplus C^\bullet_{\{0\}}[-1] \right)$ is $\mathbb{C} \oplus \mathbb{C}[-1]$

so base change with all stars fails. Morally speaking, without shrieks we are ignoring what the neighborhood of the fiber looks like.