274 Microlocal Geometry, Lecture 11

David Nadler
Notes by Qiaochu Yuan

Fall 2013

11 Constructible sheaves

Last time we introduced two perspectives on local systems on a smooth manifold M. On the one hand, we can think of it as a full subcategory of the category of sheaves of chain complexes on M. On the other hand, we can think of it as a functor from the fundamental ∞-groupoid (e.g. the simplicial set) of M to chain complexes. Roughly speaking these are analogous to vector bundles with flat connections.

Figure 1: Two views of local systems.

Example Singular cochains $C^\bullet_M$ is a local system. On a smooth manifold this is quasi-isomorphic to the de Rham complex $\Omega^\bullet_M$. The Poincaré lemma tells us that a small ball gets assigned the constant sheaf, so these local systems are in turn quasi-isomorphic to the sheaf of locally constant functions. We can think of singular cochains and the de Rham complex as resolutions of the constant sheaf.

Example Borel-Moore chains $C^{BM}_{M,-\bullet}$ assigns to an open set singular chains, not necessarily of compact support. A small ball gets assigned the orientation sheaf shifted by the dimension of M. This has something to do with Poincaré duality.

Definition Let X be a reasonable space and S a fixed stratification. *Constructible sheaves* $\mathrm{Sh}_S(X)$ on X with respect to the stratification S is the dg-category of sheaves of $\mathbb{C}$-cochain complexes with S-constructible cohomology. This means that

1. Small balls get assigned cochain complexes with finite-dimensional cohomology, and

2. If U is a neighborhood of a point x and $V \subseteq U$ is a neighborhood of another point y in the same stratum as x, then the restriction map is a quasi-isomorphism.

This definition can also be reformulated simplicially using the notion of an exit path ∞-category. Here we only allow 1-simplices that can move to less singular but not to more singular strata, and the condition on higher simplices is analogous but somewhat complicated.

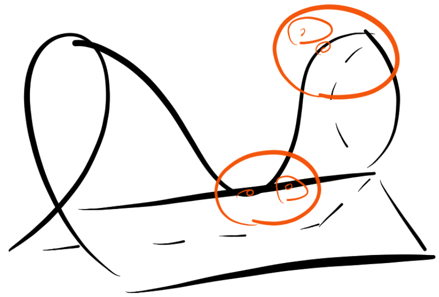

Figure 2: Two restriction maps, one of which must be a quasi-isomorphism and one of which needn't be.

Example Consider $\mathbb{R}$ with a stratification consisting of a single singular point. A constructible sheaf assigns to the singular point a chain complex F_0, assigns to the negative part another chain complex F_-, and assigns to the positive part another chain complex F_+. The exit paths out of the singular point give maps $F_- \leftarrow F_0 \rightarrow F_+$.

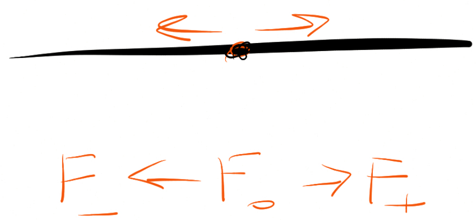

Figure 3: Constructible sheaves on $\mathbb{R}$ with a singular point.

Example Consider $\mathbb{R}^2$ with a stratification consisting of a single singular point. A constructible assigns to the singular point a chain complex F_0 and a ball in the complement another chain complex F_1. The exit path out of the singular point gives a map $r : F_0 \rightarrow F_1$. We can also consider monodromy of the ball around the singular point, which gives a map $m : F_1 \rightarrow F_1$. Finally, there is a map $h : F_1 \rightarrow F_1$ of degree -1 whose boundary exhibits a homotopy between r and $m \circ r$.

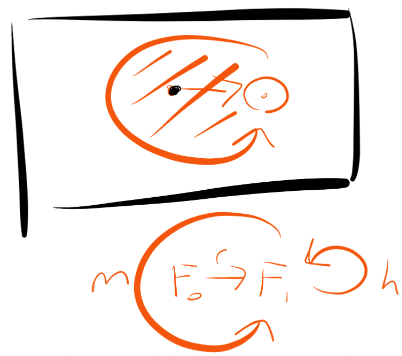

Figure 4: Constructible sheaves on $\mathbb{R}^2$ with a singular point.

Suppose a stratification S *refines* another stratification T in the sense that every stratum of T is a union of strata of S. Then there is a natural inclusion $\mathrm{Sh}_S(X) \supseteq \mathrm{Sh}_T(X)$.

Definition *Constructible sheaves* $\mathrm{Sh}(X)$ are the union of $\mathrm{Sh}_S(X)$ over all (reasonable) stratifications of X.

In other words, a sheaf is constructible if it is constructible with respect to some stratification. In particular, any local system on X is constructible.

If F is a constructible sheaf, then restricted to each stratum S_α it is a local system $F|_{S_\alpha}$. Can we reconstitute F from this data?

Let $i : Y \to X$ be a reasonable closed subset of X and let $j : U \to X$ be the inclusion of its complement. Then there is a distinguished triangle of sheaves

$$i_! i^! F \to F \to j_* j^* F \xrightarrow{[1]} . \tag{1}$$

(We may need to refine the strata and we will discuss the operations above.) This is what Grothendieck calls recollement. We can think of this as saying that F is (quasi-isomorphic to) the cone over a certain map $f : j_* j^* F[-1] \to i_! i^! F$.

To explain the symbols, j^* is the restriction map $\mathrm{Sh}(X) \to \mathrm{Sh}(U)$. This map has a right adjoint (in a suitable homotopical sense) $j_* : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ such that

$$(j_* G)(B) = G(B \cap U). \tag{2}$$

The composition $j_* j^*$ ignores things that are not happening on U. The unit of the adjunction is a natural map $F \to j_* j^* F$ given by the restriction map $F(B) \to F(B \cap U)$.

$i_! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is defined in the same way as j_* in this case. This map has a right adjoint (in a suitable homotopical sense) $i^! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ defined as follows. Think of Y as being surrounded by a tube. Given an open set V in Y we can consider the tube T living around this open set, and $(i^* F)(V)$ is sections on T relative to its boundary ∂T (by which we mean the boundary of the long part of the tube rather than its ends). We can think of this as the cone of the restriction map $F(T) \to F(\partial T)$ (where if desired we can replace ∂T with a thickening of it). This is a derived version of sections living on Y.

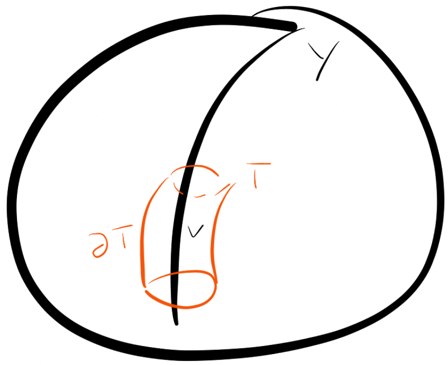

Figure 5: A tube.

The counit of this adjunction is a natural map $i_! i^! F \to F$ sending sections supported along Y into all sections. Together with the unit of the adjunction above we get our distinguished triangle.

Example Let $Y \subseteq X$ be a submanifold of a manifold and let $F = C_X^\bullet$ be cochains. Then the distinguished triangle is

$$C_Y^\bullet \otimes \mathrm{or}_{Y/X}[\mathrm{codim}_X(Y)] \to C_X^\bullet \to C_U^\bullet \xrightarrow{[1]} . \tag{3}$$

This is because, when interpreting a cochain on Y as a cochain on X, we need to keep track of codimension and coorientation.

Example Let $X = S^2$, let Y be a point, and let U be its complement. Let L be a local system on X. Then by the above, the distinguished triangle is

$$L|_Y \otimes \mathrm{or}[-2] \to L \to L|_U \xrightarrow{[1]} . \tag{4}$$

Call Y the point 0 and call another point the point 1. Then $L|_U$ is just $L|_1$. Then the boundary map in our distinguished triangle is a map $L|_1 \to L|_0 \otimes \mathrm{or}[-2]$ which is the homotopy we were looking for earlier.

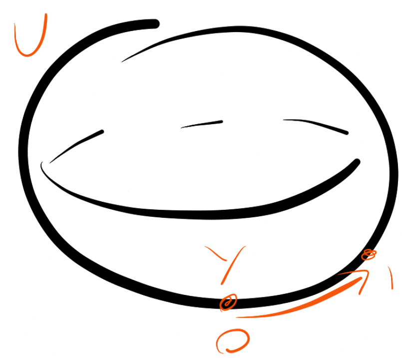

Figure 6: Constructible sheaves on S^2.