274 Microlocal Geometry, Lecture 10

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10 Local systems

Everything is with coefficients in $\mathbb{C}$ as usual.

Definition The category Sh(pt) of (complexes of) *constructible sheaves* on a point is the dg-category of chain complexes of $\mathbb{C}$-vector spaces with finite-dimensional cohomology localized at quasi-isomorphisms.

(Finite-dimensional cohomology means bounded and finite-dimensional in each degree.)

Our chain complexes will be cohomological, so with differentials

$$... \xrightarrow{d} F^{-1} \xrightarrow{d} F^0 \xrightarrow{d} F^1 \xrightarrow{d} ... \tag{1}$$

of degree 1. This category is enriched in chain complexes, hence a dg-category, because we can take the Hom complex. A quasi-isomorphism is a morphism of chain complexes which induces isomorphisms on homology. Localizing at quasi-isomorphisms means that we want to add formal inverses to all quasi-isomorphisms. Generally we only care about chain complexes up to quasi-isomorphism (this is stronger than passing to cohomology). This story is a more modern way of talking about triangulated categories.

(Keller has an ICM address on dg-categories which is a nice reference.)

Definition Let M be a manifold. The category Sh(M) of (complexes of) *constructible sheaves* on M is the dg-category of (derived) $\mathbb{C}$-local systems on M.

The above definition requires elaboration. The objects are presheaves $U \mapsto F(U)$ of chain complexes on M. We want the sheaf axioms to be satisfied, but in a more sophisticated sense; we need to consider all intersections rather than just intersections of pairs of open subsets.

Why do we need to do this? Associate to M the simplicial set Simp(M) of simplices in M. The idea is that we want to assign to every 0-simplex a chain complex F_{σ_0}, to every 1-simplex a quasi-isomorphism of chain complex, to every 2-simplex a chain homotopy between quasi-isomorphisms, etc. Because chain complexes have all higher morphisms we cannot stop here as we could with sheaves of abelian groups.

(Abstractly we are writing down an ∞-functor from some model of the fundamental ∞-groupoid of M to chain complexes.)

Constructibility means two things. First, whatever we assign to balls (contractible neighborhoods) is something with finite-dimensional cohomology. Second, the restriction map from a ball to a smaller ball is a quasi-isomorphism. This is enough to get the simplicial picture above.

Example Let $M = S^1$. We'll pick the CW-decomposition with a 0-cell and a 1-cell. A (derived? ∞?) local system assigns a chain complex F to the 0-cell and a quasi-isomorphism $m : F \to F$ to the 1-cell. Since we're working over $\mathbb{C}$ a complex with finite-dimensional cohomology is quasi-isomorphic to its cohomology, so we just have an automorphism of a finite-dimensional graded vector space.

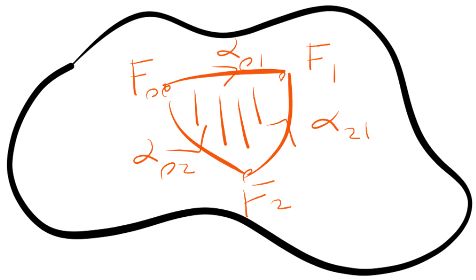

Figure 1: Data assigned to simplices.

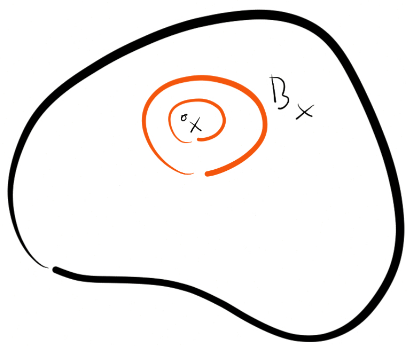

Figure 2: Restriction from balls to small balls.

Example Let $M = S^2$. Note that local systems on S^2 in the ordinary sense are just vector spaces, since $\pi_1(S^2)$ is trivial. But local systems in our sense are more interesting.

We'll pick the CW-decomposition with a 0-cell and a 2-cell. A local system assigns a chain complex F to the 0-cell and a map $m_{-1} : F \to F$ of degree -1 to the 2-cell (higher monodromy). How should we think about this map?

Let F be $\mathbb{C}$ in degrees 0 and 1 and zero otherwise. We can take $m_{-1} = 0$, which is some kind of trivial monodromy, or we can take m_{-1} to be an isomorphism from degree 0 to degree -1, which is interesting monodromy.

We can construct these things geometrically as follows. There is a projection map $\pi : S^1 \times S^2 \to S^2$, and we can take $F(U)$ to be cochains on $\pi^{-1}(U)$. This gives us trivial monodromy. We can also take the Hopf fibration $\pi : S^3 \to S^2$ and again take cochains on $\pi^{-1}(U)$. This gives us interesting monodromy.

One way to see this monodromy is to attempt to trivialize the Hopf fibration. Conflating

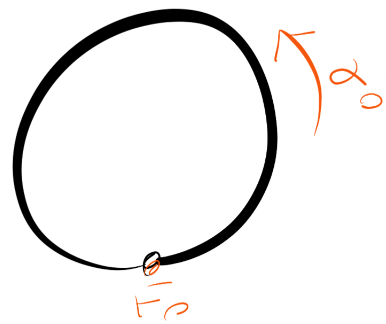

Figure 3: Local systems on a circle.

the Hopf fibration with the fibration $\mathbb{RP}^3 \to S^2$, which is the unit tangent bundle of S^2, we can attempt to write down a unit vector field that does not vanish away from some point. When we try to do this the unit vectors we assign to small neighborhoods of the point wind around the point twice.

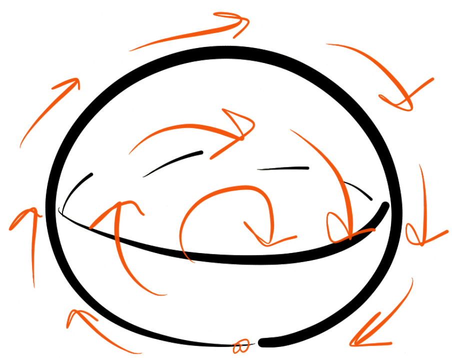

Figure 4: Attempting to write down a nonvanishing unit vector field.