# 274 Microlocal Geometry, Lecture 1

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Fall 2013

# 1 Introduction

Let $X$ be a smooth manifold. Doing local geometry means that what you're interested in depends only on a neighborhood of a point, or perhaps that you're interested in something like cohomology that has interesting local-to-global behavior. Doing microlocal geometry means that instead of studying $X$ we study its cotangent bundle $M = T^*(X)$, so that the quantities we discuss are associated not only to a point but to a point and a cotangent vector.

$M = T^*(X)$ is naturally a symplectic manifold, and the natural setting for many of the things you'd like to do to $T^*(X)$ can be done on any symplectic manifold. So microlocal geometry can be thought of as polarized symplectic geometry (a polarization is an identification of a symplectic manifold with a cotangent bundle).

## 1.1 Reasonable spaces

What is a reasonable space? (What Grothendieck called tame topology.) Roughly speaking we want a space which is locally described by a finite amount of data. What is an example of an unreasonable space? Take, for example $0 \cup \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\} \subset \mathbb{R}$. This space is not locally contractible; that's a reasonable requirement.

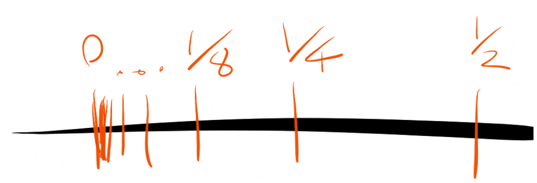


Figure 1: Infinitely many points and a limit point.

An example of an unreasonable subspace of a space is a spiral in $\mathbb{R}^2$. This is unreasonable because its intersection with the $y$-axis, say, is the space above, which is unreasonable.

Yet another unreasonable space is infinitely many distinct lines passing through the origin in $\mathbb{R}^2$ (in particular this is also not locally contractible).

So what is a reasonable space?

**Definition** An *$n$-step stratified space* $X$ is a space with a filtration $X_0 \subseteq X_1 \subseteq ... \subseteq X_n = X$ by closed subspaces such that

1. $S_i = X_i \setminus X_{i-1}$ is an $i$-manifold (the $i^{th}$ *stratum*), and

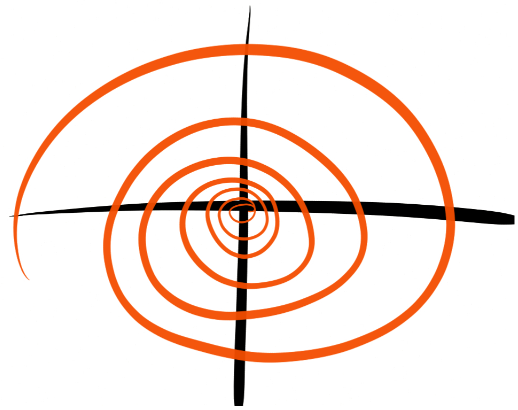

Figure 2: An infinite spiral.

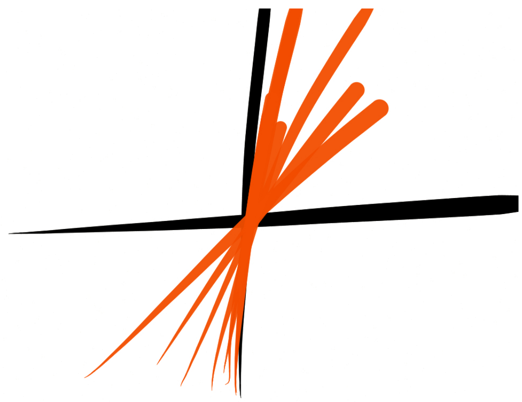

Figure 3: Infinitely many lines through the origin.

2. for all $x \in S_i$, there exists a neighborhood $x \in U \subseteq X$ and a filtration-preserving homeomorphism

$$U \cong \mathbb{R}^i \times \mathrm{Cone}(L_x) \tag{1}$$

where $L_x$ (the *link* of $x$) is a compact $(n-i-1)$-step stratified space.

Here $\mathrm{Cone}(Y)$ is the space obtained by gluing $Y \times [0, \infty)$ to a point along $Y \times \{0\}$. The cone of a filtered space acquires a filtration.

**Exercise 1.1.** *Condition 2 implies condition 1.*

**Example** A 0-step stratified space is a collection of points.

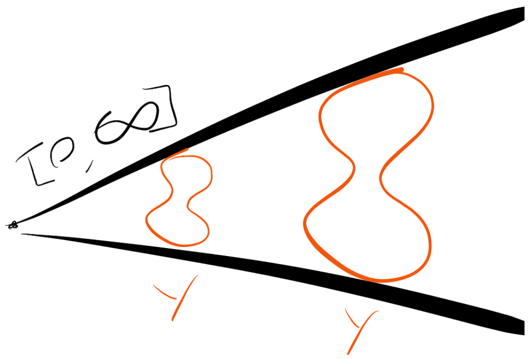


Figure 4: A picture of a cone.

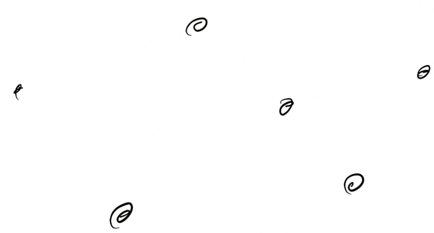

Figure 5: A collection of points.

**Example** A 1-step stratified space is in particular a space containing a collection of points such that their complement is a 1-manifold. Each point in $S_1$ must have a neighborhood homeomorphic to $\mathbb{R}$, and each point in $S_0$ must have a neighborhood homeomorphic to the cone on a finite collection of points. Hence we get locally finite graphs, possibly with isolated loops not attached to a vertex.

**Example** A famous example when $n = 2$ is called Whitney's umbrella. It consists of the union of a line and a 2-manifold, but the naive stratification one attempts to build this way fails to be a stratification because one point has unusual neighborhoods.

**Example** Another famous example when $n = 2$ is called Whitney's cusp. Again, the naive stratification fails to be a stratification because one point has unusual neighborhoods.

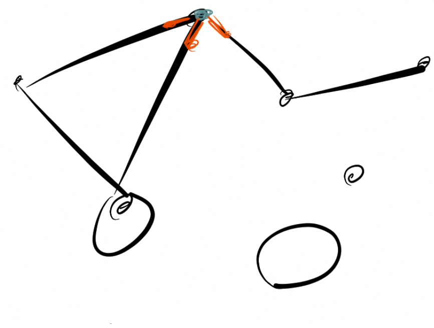

Figure 6: A locally finite graph.

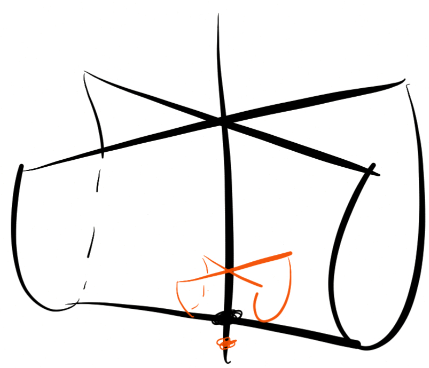

Figure 7: The Whitney umbrella and one of its unusual neighborhoods.

**Exercise 1.2.** *Let $x \in S_i$ and suppose we regard it as its own stratum. Then its link $L'_x$, when regarded as its own stratum, is the join $S^{i-1} * L_x$. In particular, if $i = 1$ then the join with $S^0$ (two points) is the suspension.*

**Example** A (locally finite) simplicial complex with at most $n$-simplices is an $n$-step stratified space where the $i^{th}$ part of the filtration is the $i$-skeleton. This is because for any point $x$, which lies in the interior of some simplex $\sigma_i$, we can take the neighborhood $U$ to be $\text{Star}(\sigma_i)$ and we can take the link to be the link in the simplicial sense.

We can think of stratified spaces as simplicial complexes where the simplices are allowed to be manifolds.

**Exercise 1.3.** *Let $X$ be a stratified space. Then the group of stratified homeomorphisms of $X$ acts transitively on every path component of every stratum.*

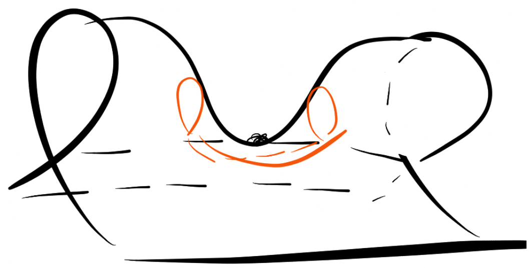

Figure 8: The Whitney cusp and one of its unusual neighborhoods.

**Non-Example** By contrast, most CW-complexes are not stratified spaces with their natural filtration. One problem is that we can write down CW-complexes with strata not satisfying the above condition.

We could call an $n$-step stratified space an $n$-dimensional stratified space, but then we should also require that $X = \overline{S_n}$.