256B Algebraic Geometry

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1 Vector bundles on the projective line

This semester we will be focusing on coherent sheaves on smooth projective complex varieties. The organizing framework for this class will be a 2-dimensional topological field theory called the B-model. Topics will include

1. Vector bundles and coherent sheaves
2. Cohomology, derived categories, and derived functors (in the differential graded setting)
3. Grothendieck-Serre duality
4. Reconstruction theorems (Bondal-Orlov, Tannaka, Gabriel)
5. Hochschild homology, Chern classes, Grothendieck-Riemann-Roch

For now we'll introduce enough background to talk about vector bundles on $\mathbb{P}^1$. We'll regard varieties as subsets of $\mathbb{P}^N$ for some N. Projective will mean that we look at closed subsets (with respect to the Zariski topology). The reason is that if $p : X \to \text{pt}$ is the unique map from such a subset X to a point, then we can (derived) push forward a bounded complex of coherent sheaves M on X to a bounded complex of coherent sheaves on a point $Rp_*(M)$.

Smooth will mean the following. If $x \in X$ is a point, then locally x is cut out by a maximal ideal m_x of functions vanishing on x. Smooth means that $\dim m_x/m_x^2 = \dim X$. (In general it may be bigger.) Intuitively it means that locally at x the variety X looks like a manifold, and one way to make this precise is that the completion of the local ring at x is isomorphic to a power series ring $\mathbb{C}[[x_1, ...x_n]]$; this is the ring where Taylor series expansions live.

If $i : x \to X$ is the inclusion of a point, we can take a (derived) pullback of a bounded complex of coherent sheaves M on X to x, and we want the result to again be a bounded complex of coherent sheaves $Li_x^*(M)$. This is true if and only if X is smooth at x.

Vector bundles can be approached from both an algebraic and a geometric perspective. From a geometric perspective, recall that an n-dimensional vector bundle $\pi : E \to X$ on a variety X is a variety E which is in some sense a vector space internal to the category of varieties over X. This includes the data of an addition map

$$+ : E \times_X E \to E \tag{1}$$

and a scalar multiplication map

$$\cdot : (\mathbb{C} \times X) \times_X E \to E \tag{2}$$

which should satisfy the vector space axioms. We also want to be able to specify some n-dimensionality and some local triviality. In algebraic geometry there are various notions of local triviality. The only one currently available is local triviality over Zariski-open subsets.

More algebraically, we can tank about a rank-n locally free $\mathcal{O}_X$-module. X is a locally ringed space, $\mathcal{O}_X$ is its sheaf of rings, and an $\mathcal{O}_X$-module is a sheaf which is a module over $\mathcal{O}_X$ in a suitable sense. Another way to say locally free is projective.

Given a vector bundle $\pi : E \to X$ in the geometric sense, we can pass to the $\mathcal{O}_X$-module $\mathcal{E}$ of sections of π, and this gives a vector bundle in the algebraic sense. Over an open set U such that the vector bundle is trivial, a section is just a collection of n functions.

Conversely, given a vector bundle $\mathcal{E}$ in the algebraic sense, the corresponding space E ought to be the relative spectrum of $\mathrm{Sym}(\mathcal{E}^\vee)$ (which is a sheaf of $\mathcal{O}_X$-algebras).

Exercise 1.1. *Check this.*

Alternatively, we can think in terms of transition functions. If U_α is an open cover of X such that the vector bundle is trivial over all U_α, then we get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$. These functions satisfy a cocycle condition, and we can use them to glue together E. The upshot of this is that n-dimensional vector bundles are classified by nonabelian cohomology $H^1(X, \mathrm{GL}_n(\mathbb{C}))$.

When $n = 1$ we talk about line bundles or invertible sheaves. Invertible sheaves are those which are invertible with respect to the tensor product. The set of isomorphism classes of line bundles on X is denoted by $\mathrm{Pic}(X)$ (the Picard group); it forms an abelian group under tensor product and dual.

Example Vector bundles on a point are vector spaces. The Picard group of a point is trivial.

Exercise 1.2. *Show that Pic*$(\mathbb{P}^1) \cong \mathbb{Z}$.

Recall that $\mathbb{P}^1$ parameterizes lines through the origin in $\mathbb{A}^2$. It is equipped with a canonical line bundle $\mathcal{O}(-1)$ whose fiber over a point is the line it parameterizes. The corresponding total space is denoted L_{-1}. This bundle should go to $-1 \in \mathbb{Z}$. The anticanonical line bundle $\mathcal{O}(1)$ is its dual, which should go to $1 \in \mathbb{Z}$. The corresponding total space is denoted L_1. In general, L_k denotes the line bundle corresponding to $k \in \mathbb{Z}$.

Theorem 1.3. *(Grothendieck) If $\pi : E \to \mathbb{P}^1$ is a rank-n vector bundle over $\mathbb{P}^1$, then E is isomorphic to a direct sum of line bundles*

$$E \cong L_{k_1} \oplus ... \oplus L_{k_n} \tag{3}$$

where, if we arrange $k_1 \geq ... \geq k_n$, then the k_i are unique.

Proof. (Sketch) Think of $\mathbb{P}^1$ as a union of two copies $\mathbb{A}^1_0, \mathbb{A}^1_\infty$ of $\mathbb{A}^1$ over $\mathbb{G}_m \cong \mathbb{C}^*$. We will use the transition function point of view, thinking of a vector bundle over $\mathbb{P}^1$ as glued together from two vector bundles over $\mathbb{A}^1$ by a transition function.

Lemma 1.4. *Every vector bundle over $\mathbb{A}^1$ is trivial.*

Proof. The ring of functions on $\mathbb{A}^1$ is a principal ideal domain, so locally free of finite rank implies free. □

In general there are plenty of nontrivial line bundles on affine varieties, e.g. an elliptic curve minus a point. However, by a difficult theorem of Quillen and Suslin, vector bundles on $\mathbb{A}^n$ are trivial.

In any case, if E is a vector bundle over $\mathbb{P}^1$, then by restricting to $\mathbb{A}^1_0$ and $\mathbb{A}^1_\infty$ we get two trivializations α_0, α_∞ and a transition function $g_{0\infty} : \mathcal{G}_m \to \text{GL}_n(\mathbb{C})$. Other choices of trivializations are given by maps

$$\alpha_0' = g_0 \circ \alpha_0, \alpha_\infty' = g_\infty \circ \alpha_\infty \tag{4}$$

where $g_0 : \mathbb{A}^1_0 \to \text{GL}_n(\mathbb{C})$ and $g_\infty : \mathbb{A}^1_\infty \to \text{GL}_n(\mathbb{C})$ are two functions. This gives

$$g_{0\infty}' = g_\infty g_{0\infty} g_0^{-1} \tag{5}$$

so our nonabelian cohomology is given by equivalence classes of maps $\mathcal{G}_m \to \text{GL}_n(\mathbb{C})$ up to the above equivalence relation. We can equivalently think of such maps as elements of $\text{GL}_n(\mathbb{C}[t, t^{-1}])$. Then $g_0 \in \text{GL}_n(\mathbb{C}[t])$ and $g_1 \in \text{GL}_n(\mathbb{C}[t^{-1}])$.

Thus isomorphism classes of n-dimensional vector bundles on $\mathbb{P}^1$ can be naturally identified with double cosets $\mathrm{GL}_n(\mathbb{C}[t^{-1}])\backslash\mathrm{GL}_n(\mathbb{C}[t,t^{-1}])/\mathrm{GL}_n(\mathbb{C}[t])$. The rest of the proof consists of the following exercise. □

Exercise 1.5. *(Birkhoff factorization) Each double coset has a unique representative with diagonal elements* $t^{k_1},...t^{k_n}$ *up to permuting the diagonal entries.*

Example $\mathcal{O}_{\mathbb{P}^1}$ is the trivial line bundle. It is specified by a 1×1 transition matrix, and any nonzero constant will do.

Example L_{taut} is the tautological line bundle. It is specified by the 1×1 transition matrix t^{-1}. To see this, we trivialize the line bundle over $\mathbb{A}^1_0 = \{[1;t] | t\in\mathbb{C}\}$ and $\mathbb{A}^1_\infty = \{[w;1] | w\in\mathbb{C}\}$. On the intersection these two coordinates are related by $w=t^{-1}$. To trivialize the bundle over $\mathbb{A}^1_0$ we need to pick a vector in each line, and one way to do this is to intersect the line with $x_0=1$. This gives the vector $(1,t)$. Similarly, trivializing the bundle over $\mathbb{A}^1_\infty$ gives the family of vectors $(w,1)$.

If $[x_0;x_1]\in\mathbb{G}_m$ lies in the intersection, then the first trivialization sends a point (y_0,y_1) in the line over $[x_0;x_1]$ to $c(1,t)$ on the one hand, where $t=\frac{x_1}{x_0}$, and to $d(t^{-1},1)$ on the other hand, where $t^{-1}=w=\frac{x_0}{x_1}$. This gives $c=dt^{-1}$, so the transition matrix is t^{-1} as desired.

Example $K_{\mathbb{P}^1}$ is the cotangent bundle (whose sections are 1-forms). There is a global 1-form which in t-coordinates looks like dt and which in w-coordinates looks like $-\frac{dw}{w^2}$. Hence the transition matrix is t^{-2}.

Example $T_{\mathbb{P}^1}$ is the tangent bundle (whose sections are vector fields). This is the dual to the cotangent bundle, and taking duals corresponds to inverting the transition matrix, hence the transition matrix is t^2.

Exercise 1.6. *Calculate the rank-2 vector bundles given by*

$$g=\begin{bmatrix} t^{-1} & 1 \\ 0 & t \end{bmatrix} \tag{6}$$

and

$$g=\begin{bmatrix} t & 1 \\ 0 & t^{-1} \end{bmatrix}. \tag{7}$$

Definition (Ad hoc) let L be a line bundle / invertible sheaf (which we will denote by $\mathcal{L}$) on $\mathbb{P}^1$. We define two cohomology groups $H^0(\mathbb{P}^1, \mathcal{L})$ and $H^1(\mathbb{P}^1, \mathcal{L})$ (it will be a theorem later that there is no higher cohomology) by writing down a chain complex, the Cech complex

$$C^0 \xrightarrow{\delta} C^1. \tag{8}$$

The first term C^0 is just the direct sum of the space of sections on $\mathbb{A}^1_0$ and $\mathbb{A}^1_\infty$. The second term C^1 is the space of sections on $\mathbb{G}_m$. The map between them is

$$\delta : (\sigma_0, \sigma_\infty) \mapsto \sigma_0 - \sigma_\infty. \tag{9}$$

Hence $H^0 = \ker(\delta)$ and $H^1 = \operatorname{coker}(\delta)$.

To compute these, note that the first component of the first term is a free module of rank 1 over the space of functions on $\mathbb{A}^1_0$, or $\mathbb{C}[t]$. We will write its generator as e_0. The second component is a free module of rank 1 over the space of functions on $\mathbb{A}^1_\infty$, or $\mathbb{C}[t^{-1}] = \mathbb{C}[w]$. We will write its generator as e_∞. The second term is a free module of rank 1 over the space of functions on $\mathbb{G}_m$, or $\mathbb{C}[t, t^{-1}]$. We will write its generator as e. The structure of the line bundle $\mathcal{L}$ determines what the differential δ looks like.

We will choose a normalization so that e_0 and e are both the function that we want to call 1 There is therefore a restriction map r_0 from $\mathbb{C}[t]$ to $\mathbb{C}[t, t^{-1}]$, which is the inclusion, and also a second restriction map r_∞.

Exercise 1.7. *Suppose $\mathcal{L}$ is the line bundle corresponding to $k \in \mathbb{Z}$. Then $r_\infty(e_\infty) = t^k e$.*

It follows from the definition that $H^0(\mathbb{P}^1, \mathcal{L})$ is just the space of global sections $\Gamma(\mathbb{P}^1, \mathcal{L})$ (since it consists of pairs of sections on the affine slices that agree on their intersection). The first cohomology describes sections on $\mathbb{G}_m$ modulo sections coming from the affine slices.

Exercise 1.8. *Interpret the cohomology groups of the list of line bundles above in terms of more classical data.*

Define the Euler characteristic of a line bundle to be $\chi(\mathbb{P}^1, \mathcal{L}_k) = \dim H^0 - \dim H^1$.

Theorem 1.9. *(Riemann-Roch for $\mathbb{P}^1$) $\chi(\mathbb{P}^1, \mathcal{L}_k) = 1 + k$.*

The second term k is the first Chern class $c_1(\mathcal{L})$ (and the first term is what could be called the zeroth Chern class $c_0(\mathcal{L})$). This theorem was further generalized by Grothendieck and Hirzebruch.

Theorem 1.10. *(Serre duality for $\mathbb{P}^1$) There is a nondegenerate pairing*

$$H^0(\mathbb{P}^1, \mathcal{L}_k) \otimes H^1(\mathbb{P}^1, \mathcal{L}_k^\vee \otimes K_{\mathbb{P}^1}) \to \mathbb{C}. \tag{10}$$

This theorem was further generalized by Grothendieck. It may be regarded as an analogue of Poincare duality.

Another important story in 20th-century mathematics is Hodge-de Rham theory. The topological story is that the topological cohomology of the Riemann sphere has dimension 1 in degrees 0 and 2 and no other degree.

Theorem 1.11. *(Hodge-de Rham for $\mathbb{P}^1$) There is a canonical identification*

$$H^\bullet_{top}(\mathbb{P}^1, \mathbb{C}) \cong H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, K_{\mathbb{P}^1}). \tag{11}$$

Very vague observation: $\mathbb{P}^1$ is not affine. If it were affine, then the line bundle $\mathcal{O}_{\mathbb{P}^1}$ would know everything (in the sense that its endomorphism ring would be the ring of global sections). Instead, there are two maps $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}$, and this diagram knows everything.

A Tannakian comment: $\mathbb{P}^1$ is not affine, so it doesn't have many global functions, hence it can't be reconstructed for them. However, it turns out that points of $\mathbb{P}^1$ can be reconstructed in terms of tensor functors $\text{Vect}_{\mathbb{P}^1} \to \text{Vect}_{\text{pt}}$.

In other words, we will take the point of view that for projective varieties, the correct replacement of the ring of functions is the category of coherent sheaves.

2 Review of sheaves

Let X be a topological space (e.g. a complex variety with the Zariski topology) regarded as a category whose objects are the open subsets of X and whose morphisms are the inclusions.

Definition Let C be a category. A *presheaf on X with values in C* on X is a functor $X^{op} \to C$.

In other words, it consists of a collection of objects $F(U)$ of C for every open subset of X and a restriction map $\rho_{UV} : F(U) \to F(V)$ for every inclusion $V \subseteq U$ which is compatible with composition of inclusions. A morphism of presheaves is a natural transformation, and this defines the category of presheaves on X with values in C.

Definition Let F be a presheaf (where C is nice, e.g. has limits). F is a *sheaf* if it satisfies the following additional property: if U is an open set and U_α an open cover of U, then $F(U)$ is the equalizer of the parallel pair of arrows

$$\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \cap U_\beta) \tag{12}$$

consisting of the two possible products of inclusions.

Example Consider the problem of solving an ODE such as $\left(\partial_z - \frac{1}{2z}\right) f(z) = 0$ on $\mathbb{C}^*$. Locally the solutions are given by the branches of the square root function. However, analytic continuation around the origin switches the two branches. This behavior is captured by the statement that there is a nontrivial sheaf of solutions; it looks like the constant sheaf $\mathbb{C}$ locally but not globally.

Definition Let $f : X \to Y$ be a continuous map of topological spaces. This induces a functor $f^{-1} : Y^{top} \to X^{top}$ assigning preimages. The composition of this functor with a sheaf on X is a sheaf $f_*(F)$ on Y, the *pushforward*.

Example There is a natural map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The pushforward of the structure sheaf turns out to be $\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(k)$.

Example There is a unique map $\mathbb{P}^n \to \text{pt}$, and the pushforward of the structure sheaf is the constant sheaf $\mathbb{C}$, which is the structure sheaf of a point.

Exercise 2.1. *If X is a closed projective variety, then the pushforward of the structure sheaf along the unique map to a point (global sections) is the constant sheaf $\mathbb{C}$.*

Proposition 2.2. *If C is a nice category, the forgetful functor $Sh(X) \to Psh(X)$ has a left adjoint called* sheafification *$F \mapsto F^+$.*

Sheafification is defined as follows for a nice concrete category C: for each $x \in X$, we define the stalk F_x as the colimit of the $F(U)$ over all open sets U containing x. We then define $F^+(U)$ to be the set of all assignments σ_x, where $x \in U$ and $\sigma_x \in F_x$, such that for all $x \in U$ there is $x \in V \subseteq U$ and $\sigma \in F(V)$ such that σ restricts to σ_y for all $y \in V$.

Generally speaking, the category of sheaves with values in C inherits structure from C.

Example Let X be a variety and consider the category of $\mathcal{O}_X$-modules (where $\mathcal{O}_X$ is the structure sheaf). It is possible to take direct sums, tensor products, and homs of sheaves. The first operation can be done pointwise but it is necessary to sheafify before performing the second two operations.

For example, take $M = \mathcal{O}_{\mathbb{P}^1}(1)$ and $N = \mathcal{O}_{\mathbb{P}^1}(-1)$. The pointwise tensor product does not have the correct space of global sections. The correct tensor product is $\mathcal{O}_{\mathbb{P}^1}$.

As another example, the category of $\mathcal{O}_X$-modules is an abelian category, so if $f : M \to N$ is a map of $\mathcal{O}_X$-modules it has a kernel and a cokernel. The kernel does not need to be sheafified (when computed as the pointwise kernel) but the cokernel does (exercise). For example, let $f = x_0 x_1$ from $M = \mathcal{O}_{\mathbb{P}^1}$ to $N = \mathcal{O}_{\mathbb{P}^1}(2)$. This morphism does not have a kernel. Its cokernel is the direct sum of two skyscraper sheaves $\mathbb{C}_{\{0\}} \oplus \mathbb{C}_{\{\infty\}}$, hence

$$0 \to M \to N \to \mathbb{C}_{\{0\}} \oplus \mathbb{C}_{\{\infty\}} \to 0 \tag{13}$$

is a short exact sequence. Taking global sections of this sequence gives

$$0 \to 0 \to \mathbb{C} \to \mathbb{C}^2 \to 0 \tag{14}$$

which is not a short exact sequence (hence global sections is not an exact functor); it is exact on the left but not on the right (hence global sections is a left exact functor). What's missing is sheaf cohomology (the right derived functors of global sections).

In general, limits can be computed pointwise but colimits need to be sheafified.

2.1 Quasicoherent sheaves

Let $X = \text{Spec } R$ where R is a (finite type?) $\mathbb{C}$-algebra. There is a functor (localization)

$$R\text{-Mod} \ni M \to \tilde{M} \in \mathcal{O}_X\text{-Mod} \tag{15}$$

defined as follows: $\tilde{M}(U)$ consists of all compatible collections of $\sigma_p \in M_p$, where M_p is the localization, satisfying the same condition as in sheafification. There is also a functor (global sections) in the opposite direction.

Definition An $\mathcal{O}_X$-module M is *quasicoherent* if for all affine opens $U = \text{Spec } R \subset X$, the restriction $M|_U$ is the localization of an R-module.

Example Let $X = \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$ and consider the sheaf

$$F(U) = \begin{cases} \mathcal{O}_X(U) & \text{if } 0 \notin U \\ 0 & \text{if otherwise} \end{cases}. \tag{16}$$

Then this module is not quasicoherent because of the following proposition.

Proposition 2.3. *Localization and global sections induce an equivalence of categories between R-Mod and QCoh(Spec R).*

Example By the above equivalence, studying quasicoherent sheaves on $\mathbb{A}^1$ is equivalent to studying $\mathbb{C}[t]$-modules. This is equivalently the category of $\mathbb{C}$-vector spaces equipped with an endomorphism. For example, the module $\mathbb{C}[t]$ itself localizes to the structure sheaf $\mathcal{O}_{\mathbb{A}^1}$. We can partially distinguish finite-dimensional modules using the characteristic polynomial. It will turn out that in this case the localization functor computes Jordan normal form in the following sense.

Let V be a $\mathbb{C}[t]$-module. First, assume it has a single Jordan block with eigenvalue λ. Suppose that we localize V away from λ; that is, we compute $V \otimes_{\mathbb{C}[t]} \mathbb{C}[t, (t-\lambda)^{-1}]$. Since $t - \lambda$ acts nilpotently on V, inverting it kills V. Hence V is zero away from λ. However, it has interesting structure near λ. For example, if $V \cong \mathbb{C}[t]/(t-\lambda)^2$, then it is still supported only at λ, but it is not just the skyscraper $\mathbb{C}[t]/(t-\lambda)$.

Geometrically, then, Jordan normal form corresponds to thinking of a (finite-dimensional) quasicoherent sheaf on $\mathbb{A}^1$ as a direct sum of skyscraper sheaves supported at the eigenvalues with additional structure given by the sizes of the Jordan blocks.

Exercise 2.4. *Giving a quasicoherent sheaf on X is equivalent to giving an R-module M for each affine open Spec $R \subset X$ compatible with restriction and satisfying the sheaf axiom.*

Proposition 2.5. *Suppose $f : X \to Y$ is a morphism of varieties. Then the push-forward f_* takes quasicoherent sheaves on X to quasicoherent sheaves on Y.*

Proof. By definition we have $f_*M)(U) = M(f^{-1}(U))$. To show quasicoherence if M is quasicoherent we need to show that if Spec $S = V \subset U =$ Spec R is an affine open then

$$f_*(M)(V) = f_*(M)(U) \otimes_S R. \tag{17}$$

To check this we just need to know that we can cover $f^{-1}(V)$ by finitely many affines and we can then induct on the number of such affines necessary. □

Definition Let $f : X \to Y$ be a morphism of schemes. The *pullback functor* f^* : QCoh$(Y) \to$ QCoh(X) is defined as follows. First, if M is a quasicoherent sheaf on QCoh(Y), define $f^{-1}(M)(U)$ to be the colimit over all open V containing $f(U)$ of $M(V)$. Second, define

$$f^*(M) = f^{-1}(M) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X. \tag{18}$$

Exercise 2.6. *f^{-1} is the left adjoint to pushforward of sheaves, and f^* is the left adjoint to pushforward of $\mathcal{O}$-modules (and quasicoherent sheaves when they can be pushed forward).*

Example Consider the inclusion $i : \mathbb{A}^1 \ni t \mapsto (t^2, t) \in \mathbb{A}^2$ and the projection $\pi : \mathbb{A}^2 \ni (z,t) \mapsto z \in \mathbb{A}^1$. With $f = \pi \circ i : \mathbb{A}^1 \ni t \mapsto t^2 \in \mathbb{A}^1$, let's compute the pushforward

$$\pi_* i_* \mathcal{O}_{\mathbb{A}^1} = f_* \mathcal{O}_{\mathbb{A}^1}. \tag{19}$$

This is some quasicoherent sheaf on $\mathbb{A}^1$ which is therefore completely determined by its global sections. We can compute global sections by pushing forward to a point, which gives $\mathbb{C}[z]$ as a vector space. As a module, z acts by z^2. This gives a free rank-2 module M.

This is not very interesting from the point of view of Jordan canonical form, but now we can regard the $\mathbb{A}^1$ we project onto as a parameter space and the $\mathbb{A}^1$ we map from as the space we're computing Jordan canonical forms on. To do this, we instead compute

$$i_\epsilon^* f_* \mathcal{O}_{\mathbb{A}^1} = M \otimes_R \mathbb{C}_\epsilon = M_\epsilon \tag{20}$$

where $i_\epsilon : \mathrm{pt} \to \mathbb{A}^1$ is the inclusion of the point ϵ. For fixed ϵ, M_ϵ is a 2-dimensional complex vector space. When $\epsilon \neq 0$, the endomorphism t acts by $\begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}$, but when $\epsilon = 0$, the endomorphism t acts by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Exercise 2.7. *Consider the cross X cut out by $x^2 = y^2$ in $\mathbb{A}^2$. If $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ is as above, analyze $\pi_*(\mathcal{O}_X)$. Try to find the geometry of the family of Jordan normal forms* $t = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix}$.

Example Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism. We can write $f = \frac{p}{q}$ where p, q are two polynomials with no common factors. In the special case that $f(z) = z^d$, we have $f^*(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}$ (the pullback of the structure sheaf is the structure sheaf). The pushforward is more interesting. The restriction $f_0 : \mathbb{A}^1_0 \to \mathbb{A}^1_0$ is given by $f_0(z) = z^d$ and there is another restriction $f_\infty : \mathbb{A}^1_\infty \to \mathbb{A}^1_\infty$.

The pushforward along f_0 is the module $\mathbb{C}[t]$, where t acts by t^d. This is a free module of rank d. The same thing is true of the pushforward along f_1, hence the pushforward is a vector bundle of rank d. Hence it is a direct sum of d line bundles.

Recall that we know something about the cohomology of line bundles on $\mathbb{P}^1$. If we can compute the cohomology of the pushforward, we may be able to determine the pushforward itself. So we will try to calculate

$$H^*(\mathbb{P}^1, f_*(\mathcal{O}_{\mathbb{P}^1})). \tag{21}$$

We do this by writing down the complex we wrote down previously, namely

$$\Gamma(\mathbb{A}^1_0, V) \oplus \Gamma(\mathbb{A}^1_\infty, V) \xrightarrow{r_0 - r_\infty} \Gamma(\mathbb{G}_m, V) \tag{22}$$

and computing its cohomology (where $V = f_*(\mathcal{O}_{\mathbb{P}^1})$). This gives the complex

$$\mathbb{C}[t] \oplus \mathbb{C}[t^{-1}] \xrightarrow{r_0 - r_\infty} \mathbb{C}[t, t^{-1}] \tag{23}$$

whose cohomology is $\mathbb{C}$ in degree 0 and 0 in degree 1; note that the cohomology does not notice f because 0 is sent to 0 and ∞ is sent to ∞. This identifies a unique sum of line bundles, namely $d-1$ copies of L_{-1} and one copy of L_0.

Exercise 2.8. *Compute pushforwards along more complicated maps. Compute pushforwards of more complicated vector bundles.*

2.2 The canonical bundle

On any variety there is always a natural line bundle, namely the structure sheaf. On a smooth variety X there is another natural line bundle, namely the top exterior power of the cotangent bundle, called the canonical bundle

$$\kappa_X = \Lambda^{\dim X} \Omega^1_X. \tag{24}$$

Recall that sections of the cotangent bundle are 1-forms. One way of thinking about this is as follows. If X is a variety, we can look at the diagonal embedding $\Delta : X \to X \times X$. In differential geometry, the normal bundle of X in $X \times X$ is the tangent bundle. In algebraic geometry, we instead consider the ideal sheaf $I_\Delta \subset \mathcal{O}_{X \times X}$ of functions vanishing on the diagonal, so that $\mathcal{O}_X \cong \mathcal{O}_{X\times X}/I_\Delta$ as sheaves.

We can instead look at $\mathcal{O}_{X\times X}/I_\Delta^2$. There is a sequence of sheaves on X

$$0 \to I_\Delta/I_\Delta^2 \to \mathcal{O}_{X\times X}/I_\Delta^2 \to \mathcal{O}_X \to 0. \tag{25}$$

and the leftmost term is Ω^1_X.

As practice we would like to compute the canonical bundle of $\mathbb{P}^n$. As context we want the following.

Exercise 2.9. *Show that $Pic(\mathbb{P}^n) \cong \mathbb{Z}$ with the isomorphism given by sending $\mathcal{O}_{\mathbb{P}^n}(k)$ to k.*

So the canonical bundle is $\mathcal{O}_{\mathbb{P}^n}(k)$ for some k and the problem is to compute k.

Let $X \subset \mathbb{P}^n$ be a degree-d hypersurface. Recall that sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ are homogeneous polynomials of degree d. Given such a section we can talk about the points where it vanishes, and X is the zero locus of such a section. Restriction of forms gives a map $\Omega^1_{\mathbb{P}^n} \to \Omega^1_X$ fitting into a short exact sequence of sheaves on X

$$0 \to N^\vee_{X/\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}|_X \to \Omega^1_X \to 0 \tag{26}$$

where $N^\vee_{X/\mathbb{P}^n}$ is an invertible sheaf called the conormal sheaf.

Lemma 2.10. *The conormal bundle is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^n}(-d)$ to X.*

Proof. (Intuitive sketch) Equivalently, we want to show that $N_{X/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(d)$. Recall that X is the zero section of a global section $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Roughly speaking, a section of the normal bundle of X is a way to wiggle X infinitesimally inside $\mathbb{P}^n$. We can wiggle X by wiggling σ. But the tangent space to a point in a vector space is the vector space. □

Recall that if $0 \to A \to B \to C \to 0$ is a short exact sequence, then the top exterior power of B is the top exterior power of A tensor the top exterior power of C.

Corollary 2.11. $\kappa_{\mathbb{P}^n}|_X \cong \kappa_X \otimes \mathcal{O}_{\mathbb{P}^n}(-d)|_X$.

As an application, we can compute that

$$\kappa_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \tag{27}$$

by induction using the fact that $\mathbb{P}^{n-1}$ is a linear ($d = 1$) hypersurface in $\mathbb{P}^n$.

As another application, let E be a cubic curve in $\mathbb{P}^2$. Then

$$\mathcal{O}_{\mathbb{P}^2}(-3)|_E \cong \kappa_{\mathbb{P}^2}|_E \cong \kappa_E \otimes \mathbb{O}_{\mathbb{P}^2}(-3)|_E \tag{28}$$

from which it follows that $\kappa_E \cong \mathcal{O}_E$; the canonical bundle is trivial. More generally, a variety with this property is called a Calabi-Yau variety. We can obtain such things using degree-$n+1$ hypersurfaces in $\mathbb{P}^n$.

In general, small-degree hypersurfaces look like projective space. At some point they become Calabi-Yau. Beyond that, they become general type.

2.3 More about line bundles

Let X be a smooth projective complex variety. We've discussed two line bundles, the trivial line bundle $\mathcal{O}_X$ and the canonical bundle κ_X. When $X = \mathbb{P}^n$ these generate the Picard group. Today we will describe more line bundles on curves.

First, we can think of $\mathcal{O}_X$ as sitting inside a much larger sheaf, the sheaf $\mathcal{K}_X$ of rational functions on X. This is a (locally) constant sheaf. It is also quasicoherent. We will define other line bundles as subsheaves of this giant sheaf.

Definition A *divisor* D on X is an element of the free abelian group $\text{Div}(X)$ on the (closed) points of X.

Divisors naturally occur as follows. If f is a regular function and $x \in X$ is a point, we can assign an order $\text{ord}_x(f)$ based on the x-adic valuation of f in the x-adic completion of $\mathcal{O}_{X,x}$. More generally we can do this with rational functions. Hence we can think of divisors as prescribing orders (zero / pole behavior). We can use this to assign to each divisor a line bundle, giving a map

$$\text{Div}(X) \ni D \mapsto \mathcal{O}_X(D) \in \text{Pic}(X) \tag{29}$$

defined as follows: the sections of $\mathcal{O}_X(D)$ are rational functions f such that $\text{ord}_x(f) \geq n_x$ if $D = \sum n_x x$.

Exercise 2.12. $\mathcal{O}_X(D)$ *is locally free of rank* 1.

Exercise 2.13. $\mathcal{O}_X(0) = \mathcal{O}_X$.

There is a map $\mathcal{K}_X^\times \to \text{Div}(X)$ assigning a rational function its divisor. Its image is the subgroup of *principal divisors* $\text{PDiv}(X)$.

Theorem 2.14. *There is a short exact sequence*

$$0 \to PDiv(X) \to Div(X) \to Pic(X) \to 0 \tag{30}$$

inducing an isomorphism $Div(X)/PDiv(X) \cong Pic(X)$.

Some intuition: consider dividing up a variety, in some formal sense, as the union of the formal neighborhoods of each point (spectra of the corresponding formal power series rings) and a generic point (spectra of the rational functions). On each such open set, a line bundle is trivializable. The gluing data joining together all of these trivializations gives more or less the above description (where we think of the order of a rational function at a point as an element of $\mathcal{K}_x^\times/\mathcal{O}_{X,x}^\times$).

Proof. We want to show that the map $\text{Div}(X)/\text{PDiv}(X) \to \text{Pic}(X)$ is a homomorphism, injective, and surjective. The first statement means $\mathcal{O}_X(D_1 + D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$. There is a map in the other direction given by multiplying $f_1 \in \mathcal{O}_X(D_1)$ by $f_2 \in \mathcal{O}_X(D_2)$. To check that this is an isomorphism it suffices to check on small open sets or even stalks. There is a small lemma necessary here, namely that we can find meromorphic functions with arbitrary order at a given point.

Here is a slick proof of the rest. We can think of $\text{Div}(X)$ as the sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. We also have $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$. There is a short exact sequence of sheaves

$$0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \text{Div}(X) \to 0 \tag{31}$$

and the corresponding long exact sequence in cohomology gives

$$H^1(X, \mathcal{O}_X^\times) \cong H^0(\text{Div}(X))/H^0(\mathcal{K}_X^\times) \tag{32}$$

which gives the conclusion. □

Exercise 2.15. *Let E be an elliptic curve. There is an isomorphism $\mathbb{Z} \times E \to Pic(E)$ which sends $(k, x_1) \mapsto \mathcal{O}_X(x_1 - (k+1)x_0)$ where x_0 is the identity of E.*

In particular, $\text{Pic}(E \setminus \{x_0\})$ is an affine variety with nontrivial line bundles on it.

3 Homological algebra

Pushforward and pullback do not behave quite the way we want them to; we should be taking the derived pushforward and pullback instead.

Let $f : X \to Y$ be a map of complex varieties. f determines an adjunction $(f^*, f_*) : \text{QCoh}(Y) \to \text{QCoh}(X)$. The pullback f^* is more of an algebraic operation (the interesting part is the tensor product), while the pushforward f_* is more of a topological operation.

The pullback is the left adjoint and the pushforward is the right adjoint. Left adjoints preserve colimits, hence the pullback is right exact and has left derived functors. Right adjoints preserve limits, hence the pullback is left exact and has right derived functors.

Example Let X be a point and let $Y = \mathbb{P}^1$. The subsheaf of sections of the structure sheaf on Y vanishing at X is $\mathcal{O}_{\mathbb{P}^1}(-1)$, giving a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathbb{C}_X \to 0 \tag{33}$$

of sheaves on Y. Pulling back to X gives

$$\mathbb{C} \xrightarrow{0} \mathbb{C} \to \mathbb{C} \to 0 \tag{34}$$

which is right exact but not exact.

Example Let $X = \mathbb{P}^1$ and let Y be a point. Consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{C}_{\text{pt}} \to 0 \tag{35}$$

of sheaves on X, where the first sheaf is functions with a zero of order at least 2 at a point and the second sheaf is functions with a zero of order at least 1 at a point. Pushing forward to Y means taking global sections, which gives

$$0 \to 0 \to 0 \to \mathbb{C} \tag{36}$$

which is left exact but not exact.

We will fix this failure of exactness as follows. Instead of working with sheaves, we will work with (certain) complexes of sheaves.

Definition An object P in an abelian category is *projective* if $\text{Hom}(P, -)$ is exact. Dually, an object I is *injective* if $\text{Hom}(-, I)$ is exact.

Definition Let f^* be a right exact functor. Its *left derived functor* $Lf^*(N)$ is given by the cohomology of $f^*(P^\bullet)$ where P is a *projective resolution* of N (a complex consisting of projective objects which is quasi-isomorphic to N).

Dually, we can define the right derived functors $Rf_*(M)$ by taking the cohomology of f_* applied to an injective resolution $I^\bullet$ of M.

Before we take cohomology, we have an operation that takes as input a sheaf and produces as output a complex of sheaves. This complex depends on the choice of projective resolution, although its cohomology does not. We would like to work in a setting that does not require passing to cohomology to get something that does not depend on a choice of projective resolution.

Example Consider the inclusion of a point $X = \{0\}$ into $Y = \mathbb{A}^1$ and consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{A}^1} \xrightarrow{y} \mathcal{O}_{\mathbb{A}^1} \to \mathbb{C}_0 \to 0 \tag{37}$$

or, more algebraically,

$$0 \to \mathbb{C}[y] \xrightarrow{y} \mathbb{C}[y] \to \mathbb{C}[y]/(y) \to 0. \tag{38}$$

The pullback is tensoring by $\mathbb{C}[y]/y$. We do not need to resolve $\mathbb{C}[y]$ before tensoring, but we do need to resolve $\mathbb{C}[y]/y$ (using the above short exact sequence). Applying the left derived functor now gives a long exact sequence

$$0 \to \mathbb{C}[y]/y \to \mathbb{C}[y]/y \xrightarrow{0} \mathbb{C}[y]/y \to \mathbb{C}[y]/y \to 0. \tag{39}$$

Example Consider again the map from $X = \mathbb{P}^1$ to a point Y. To compute derived pushforward we need an injective resolution. These are usually messy. One option is to take the Godement resolution, which is defined as follows.

First, the topological step. If F is a sheaf on a space X and $i_x : x \to X$ is the inclusion of a point, we can consider $(i_x)_*(i_x)^*F$, which is just the stalk F_x of F at x. There is a natural map $F \to F_x$, and for starters we can look at the map

$$F \to \prod_x F_x. \tag{40}$$

This is an injection.

Next, the algebraic step. Insert each F_x into its injective hull $I(F_x)$. This gives a map

$$F \to \prod_x I(F_x). \tag{41}$$

We can now take the cokernel of this map and repeat.

We will not do this in practice. Instead of using injective resolutions, it turns out that we can resolve by objects that have no extra derived cohomology. If i_0, i_∞ are the inclusions of the two affines into $\mathbb{P}^1$ and i is the inclusion of $\mathfrak{G}_m$, then there is a resolution of $\mathcal{O}_{\mathbb{P}^1}(-2)$ of the form

$$\mathcal{O}_{\mathbb{P}^1}(-2) \to (i_0)_*(i_0)^*\mathcal{O}_{\mathbb{P}^1}(-2) \oplus (i_\infty)_*(i_\infty)^*\mathcal{O}_{\mathbb{P}^1}(-2) \to i_*i^*\mathcal{O}_{\mathbb{P}^1}(-2) \tag{42}$$

which is essentially the Cech resolution, and computing the derived pushforward Rf_* with this resolution reproduces the Cech definition of sheaf cohomology we gave earlier.

We can in general compute the derived pushforward using Cech resolutions. Let X be a topological space, F a sheaf on X (of abelian groups, for example), and U a cover of X by open sets. The *Cech complex* is

$$\prod_\alpha F(U_\alpha) \to \prod_{\alpha,\beta} F(U_\alpha \cap U_\beta) \to \prod_{\alpha,\beta,\gamma} F(U_\alpha \cap U_\beta \cap U_\gamma) \to ... \tag{43}$$

where by convention we only consider distinct U_α and each differential is an alternating sum of inclusions. The *Cech cohomology* $H^\bullet_U(X;F)$ is the cohomology of this complex.

Specialize to the case that X is a complex variety, U is a cover by affines, and F is a quasicoherent sheaf.

Theorem 3.1. *$H^\bullet_U(X;F)$ calculates $Rf_*(F)$ where $f : X \to pt$ is the unique map to a point (sheaf cohomology).*

Exercise 3.2. *Check that $H^\bullet_U(X;F)$ does not depend on the choice of open cover U.*

Corollary 3.3. *If X is affine, then $H^i(X;F) \cong R^if_*F = 0$ for $i > 0$.*

So derived pushforward detects topology. On the other hand, derived pullback is interesting even on affines.

Serre's criterion gives a converse to the above: if $H^i(X;F) = 0$ for $i > 0$ and all $F \in \text{QCoh}(X)$, then X is affine.

Exercise 3.4. *Show that $X = \mathbb{C}^2 \setminus \{(0,0)\}$ is not affine (because $\mathcal{O}_X$ has higher cohomology).*

Corollary 3.5. *Let X be a (projective) variety with $\dim X = k$. Then $H^i(X; F) = 0$ for $i > k$.*

Proof. If X is projective, embed X into $\mathbb{P}^N$. Choose $k+1$ hyperplanes $H_0, ...H_k$ in general position such that their intersection misses X. Take the affine cover $U_i = X \cap (\mathbb{P}^N \setminus H_i)$. □

Theorem 3.6. *The cohomology of line bundles on projective space is the following:*

$$\dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{n+m}{m}, m \geq 0 \tag{44}$$

$$\dim H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{-m-1}{-n-m-1}, m \leq -n-1 \tag{45}$$

and all other n, m give zero.

Global sections are reasonable; global sections of $\mathcal{O}_{\mathbb{P}^n}(m)$ are given by homogeneous polynomials of degree m in $n+1$ variables. H^n turns out to be determined by Serre duality.

Example Consider $\mathbb{P}^2$. Describe it with homogeneous coordinates x_0, x_1, x_2 and consider the corresponding affine opens U_0, U_1, U_2, all isomorphic to $\mathbb{C}^2$. Their triple intersection is $(\mathbb{C}^*)^2$ and their double intersections are $\mathbb{C}^* \times \mathbb{C}$.

The following visualization may be helpful for computing the relevant cohomology. Consider the monoid generated by $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{x_2}{x_0}$. Its elements can be visualized as lattice points in $\mathbb{R}^2$, and collections of such points describe functions holomorphic on various pieces above.

Corollary 3.7. *Let $f : X \to Y$ be a projective morphism of complex varieties and let F be a coherent sheaf (locally finitely presented) on X. Then $R^i f_* F$ is coherent and vanishes for sufficiently large i.*

This is why we like projective things; in general, pushforwards of finite things will not be finite. For example, the pushforward of the structure sheaf from $\mathbb{A}^1$ to a point is $\mathbb{C}[t]$.

We will focus on the case $Y = \text{pt}$ (in which case X is projective and the derived pushforward is sheaf cohomology). For the proof we will need the following.

Exercise 3.8. *If F is a coherent sheaf on X and we embed X as a closed subspace of $\mathbb{P}^n$, then there is a short exact sequence*

$$0 \to K \to \mathcal{O}_{\mathbb{P}^n}(m)^{\oplus I} \to F \to 0 \tag{46}$$

where K is also coherent.

We now return to the proof.

Proof. Consider the long exact sequence associated to the above short exact sequence. The associated cohomologies vanish in sufficiently large degrees and by induction we can get finite-dimensionality. □

Definition Let F be a coherent sheaf on a projective variety X. Its *Euler characteristic* is

$$\chi(X; F) = \sum (-1)^i \dim H^i(X; F). \tag{47}$$

Proposition 3.9. *(Vaguely speaking) Let $X_U \to \mathbb{P}^n \times U$ be a U-family of projective varieties. Then $\chi(X_u; \mathcal{O}_{X_u})$ is locally constant for $u \in U$.*

Somewhat more precisely, suppose that X is a projective variety cut out by equations $f_1, ...f_k$ of degrees $m_1, ...m_k$. Then we can write a short exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(m_i) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0. \tag{48}$$

Varying X in a family X_u means letting the f_i vary algebraically depending on $u \in U$.

Example Consider the family of elliptic curves given by the projective closures of $y^2 = x^3 + ax + b$. Here $U = \mathbb{A}^2$ with coordinates a, b.

Proof. The Euler characteristic is additive in short exact sequences, hence

$$\chi(\mathbb{P}^n) = \chi(\mathcal{O}_{X_u}) + \chi(\bigoplus \mathcal{O}_{\mathbb{P}^n}(m_i)). \tag{49}$$

□

The Riemann-Roch theorem is the special case of Hirzebruch-Riemann-Roch for curves, which is in turn the special case of Grothendieck-Riemann-Roch for maps to a point. All of these theorems involve calculating Euler characteristics.

Let C be a smooth projective curve of genus g and let L be a line bundle over C.

Theorem 3.10. *(Riemann-Roch)* $\chi(C; L) = \dim H^0(C; L) - \dim H^1(C; L) = 1 - g + \deg(L)$.

Here if $L = \mathcal{O}_C(D)$ for a divisor D then $\deg(L) = \deg(D)$. More invariantly, $\deg(L)$ is the first Chern class $c_1(L)$. The genus g may be interpreted as $\dim H^1(C; \mathcal{O}_C)$, but it is also the topological genus of C.

Roughly speaking the LHS is algebraic and the RHS is topological.

Proof. We will reduce to the case that $L = \mathcal{O}_C$. The claim is that this formula is true for $L = \mathcal{O}_C(D)$ if and only if it's true for $L = \mathcal{O}_C(D + p)$ where p is a point. Consider the short exact sequence

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + p) \to \mathbb{C}_p \to 0. \tag{50}$$

Taking Euler characteristics gives

$$\chi(C, \mathcal{O}_C(D + p)) = \chi(C, \mathcal{O}_C(D)) + 1. \tag{51}$$

This gives

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C) + \deg(D) \tag{52}$$

where $\chi(C, \mathcal{O}_C) = 1 - \dim H^1(C; \mathcal{O}_C)$ as desired. □

We now want to show that $\chi(C, \mathcal{O}_C)$ is in fact $1-g$ where g is the topological genus. For example, if E is an elliptic curve, it can be described as a degree-3 hypersurface in $\mathbb{P}^2$, giving a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_E \to 0. \tag{53}$$

The Euler characteristics are $1, 1, 0$ by additivity.

In general, we claim that a genus-g algebraic curve C can be degenerated (by moving around equations defining it suitably) into a singular curve C_{sing} with g singularities. It has a normalization which is just $\mathbb{P}^1 = \widetilde{C_{\text{sing}}}$. We have

$$\chi(C;\mathcal{O}_C) = \chi(C_{\text{sing}};\mathcal{O}_{C_{\text{sing}}} \tag{54}$$

so it suffices to compute the latter. If $\pi : \mathbb{P}^1 \to C_{\text{sing}}$ is the normalization map, we have a short exact sequence

$$0 \to \mathcal{O}_{C_{\text{sing}}} \to \pi_* \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{g} \mathbb{C}_{p_i} \to 0 \tag{55}$$

where the p_i are the nodes. (The second map here reflects the fact that C_{sing} is obtained from $\mathbb{P}^1$ by gluing g pairs of points together, and while a function on $\mathbb{P}^1$ may have different values on these pairs, a function on C_{sing} must have the same value.) Taking Euler characteristics, the result follows.

Exercise 3.11. *Compute the Cech cohomology of the structure sheaf of C_{sing}.*

4 dg-categories

We've assigned to a complex variety X its category $\text{QCoh}(X)$ of quasicoherent sheaves. In this category there's a subcategory $\text{Coh}(X)$ of coherent sheaves and a further subcategory $\text{Vect}(X)$ of vector bundles. If $f : X \to Y$ is a morphism, we've defined an adjunction (f^*, f_*) on quasicoherent sheaves, and we've also defined their derived functors Lf^*, Rf_*. The cohomologies $L^i f^*$ and $R^i f_*$ take quasicoherent sheaves to quasicoherent sheaves.

If f is Tor-finite (e.g. smooth; roughly speaking this means that fibers are smooth, and precisely it refers to the relative cotangent bundle having the expected dimension; the analogue in differential geometry is a fibration with smooth fibers) then $L^i f^*$ vanishes for large i and takes coherent sheaves to coherent sheaves. If f is proper (e.g. projective) then $R^i f^*$ has the same property.

Example Let $X = \text{pt}$ and let $Y = \{(x, y) \in \mathbb{A}^2 : xy = 0\}$. Let $f : X \to Y$ be the inclusion of the origin. We want to compute $Lf^*(\mathbb{C}_{(0,0)})$. Let $R = \mathbb{C}[x, y]/(xy)$ (so that $Y = \text{Spec } R$). We can write down a free resolution of the form

$$\cdots \to R^2 \to R^2 \to R^2 \to R \to \mathbb{C}_{(0,0)} \tag{56}$$

where the differentials alternate between sending $(1, 0)$ to $(x, 0)$ and $(0, 1)$ to $(0, y)$ and sending $(1, 0)$ to $(y, 0)$ and $(0, 1)$ to $(0, x)$, the second differential sends $(1, 0)$ to

x and $(0,1)$ to y, and the first differential sends 1 to 1. Applying f^* to the resolution gives

$$\cdots \to \mathbb{C}^2 \to \mathbb{C}^2 \to \mathbb{C}^2 \to \mathbb{C} \tag{57}$$

where all of the maps are zero. Hence f is not Tor-finite and $L^i f^*$ does not vanish in high degrees.

Example Let $f : \mathbb{A}^1 \to \text{pt}$ be the unique map. Then $Rf_*(\mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[t]$ in degree 0, which is not coherent over pt.

We want a better way to organize derived functors. If $f : X \to Y$ is a nice map, we would like to define categories $D_{\text{coh}}(X)$ so that the left and right derived functors are adjoint functors $Lf^* : D_{\text{coh}}(Y) \to D_{\text{coh}}(X)$ and $Rf_* : D_{\text{coh}}(X) \to D_{\text{coh}}(Y)$. These are going to be derived categories, and we would like to understand what they capture about varieties. (This is a higher version of assigning to X the functions on X, which can be pulled back and can occasionally be pushed forward.) If X has a reasonable canonical bundle, it turns out to be possible to recover X from $D_{\text{coh}}(X)$.

Instead of working with derived categories, though, we are going to pass directly to dg-categories.

Definition A *differential graded category* (over $\mathbb{C}$), or *dg-category*, is a category C enriched over the category $\text{Ch}(\mathbb{C}\text{-Mod})$ of chain complexes of complex vector spaces. In particular, for every $x, y \in C$ there are complex vector spaces $\text{Hom}^i(x,y)$ and differentials $d : \text{Hom}^i(x,y) \to \text{Hom}^{i+1}(x,y)$ with $d^2 = 0$, and composition is a morphism

$$\circ : \text{Hom}(x,y) \otimes \text{Hom}(y,z) \to \text{Hom}(x,z) \tag{58}$$

where $\otimes$ is the tensor product of chain complexes. A *dg-functor* between dg-categories $F : C \to D$ is a $\text{Ch}(\mathbb{C}\text{-Mod})$-enriched functor.

Definition The *homotopy category* $H(C)$ of a dg-category is the $\mathbb{C}$-linear category whose objects are the objects of C and whose morphisms are given by

$$\text{Hom}_{H(C)}(x,y) = H^0(\text{Hom}(x,y)). \tag{59}$$

Similarly we can define the graded homotopy category $H^\bullet(C)$ which keeps track of all the cohomologies.

There is an obvious notion of equivalence of dg-categories which is not the one we will use.

Definition A dg-functor $F : C \to D$ is a (weak) *equivalence* if $H(F) : H(C) \to H(D)$ is an equivalence and $H^\bullet(F) : H^\bullet(C) \to H^\bullet(D)$ is an isomorphism on morphisms (equivalently, $H^\bullet(F)$ is an equivalence).

Example A dg-category with one object is a *dg-algebra.* The endomorphism algebra of any object in a dg-category is a dg-algebra. If M is a smooth manifold, the de Rham complex $\Omega^\bullet(M)$ is naturally a dg-algebra.

Example If C is a dg-category, the opposite category C^{op} is also a dg-category. (This requires that we know that chain complexes have a symmetric monoidal structure.) Specialized to dg-algebras, we get the notion of the opposite algebra of a dg-algebra.

Exercise 4.1. *Let C be a dg-category. The dg-category of dg-functors $F : C \to$ Ch($\mathbb{C}$-Mod) is the dg-category of left dg-modules over C, and for C^{op} we get right dg-modules. The dg-category structure comes from the fact that Ch($\mathbb{C}$-Mod) is enriched over itself. This specializes to the intuitive notion of module when C has one object.*

Definition The *Yoneda embedding* is the embedding

$$C \ni x \mapsto \mathrm{Hom}(-, x) \in \mathrm{Mod}\text{-}C. \tag{60}$$

It is fully faithful.

For a dg-algebra A, this embeds A into the category of right A-modules as the automorphism dg-algebra of the right module A_A. This is a form of Cayley's theorem.

Among all dg-categories, categories of chain complexes have the following special properties:

1. We can consider direct sums and take direct summands.
2. We can shift chain complexes $C \mapsto C[k]$, where $C[k]_n = C_{n+k}$ and $d[k] = (-1)^k d$.

3. We can take cones. If $f : C \to D$ is a morphism, the cone $\mathrm{cone}(f)$ is a complex with components $\mathrm{cone}(f)_n = C_{n+1} \oplus D_n$ and differential built from f and the differentials of C and D. There is a natural inclusion $D \to \mathrm{cone}(f)$ and a natural projection $\mathrm{cone}(f) \to C[1]$; these fit into a *triangle*.

Definition The *triangulated hull* or *stabilization* C^{st} of a dg-category C is obtained from the image of the Yoneda embedding of C by adding finite direct sums, direct summands, and cones (which includes shifts).

Definition A dg-category C is (pre)*triangulated* or *stable* if the embedding $C \to C^{st}$ is an equivalence.

Exercise 4.2. *There is a dg-category C_{cone} such that a dg-functor $C_{cone} \to C$ is the same thing as a triangle in C.*

C_{cone} has three objects X, Y, Z where $\mathrm{Hom}(X, X), \mathrm{Hom}(Y, Y)$ are $\mathbb{C}$ in degree 0 and $\mathrm{Hom}(Z, Z)$ is $\mathbb{C} \oplus \mathbb{C}$ in degree 0 and $\mathbb{C}$ in degree 1 with surjective differential. $\mathrm{Hom}(X, Y)$ is $\mathbb{C}$ in degree 1 and $\mathrm{Hom}(Y, X)$ is zero. $\mathrm{Hom}(Y, Z)$ and $\mathrm{Hom}(Z, X)$ are $\mathbb{C}$ in degree 0. $\mathrm{Hom}(Z, Y)$ and $\mathrm{Hom}(X, Z)$ is $\mathbb{C}$ in degrees 0 and 1 with identity differential. The graded homotopy category of this category has no interesting compositions.

4.1 Chain complexes

Chain complexes are important; sometimes it does not suffice only to consider their homology.

Example Consider two circles S^1_a, S^1_b in S^3. We cannot detect whether they are linked using only the induced maps on homology. Nevertheless, there is an interesting map $H_1(S^1_a) \otimes H_1(S^1_b) \to H_0(S^3)$ which detects linkedness, and we need to look at chains instead of homology to see it. First consider chains $C_\bullet(S^3)$ on S^3. The circle S^1_a is trivial, but it is trivial for a reason; it is the boundary of some $D_a \in C_2(S^3)$. This is not a homology class. We can now form the intersection $D_a \cap S^1_b \in C_0(S^3)$; this is a Massey product $m(\gamma_a \otimes \gamma_b)$.

Example Consider S^1 acting on S^3 via the Hopf action (induced from the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$). This is a free action with quotient S^2. How nontrivial is this action? We can look at the induced map $H_\bullet(S^1) \otimes H_\bullet(S^3) \to H_\bullet(S^3)$ and it turns out that this map is not interesting. We can instead look at an action

$$m_1 : H_1(S^1) \otimes C_\bullet(S^3) \to C_\bullet(S^3) \tag{61}$$

on chains. Considering a point $\mathrm{pt} \in C_0(S^3)$, we get a circle $m_1(\mathrm{pt}) \in C_1(S^3)$ which bounds a disk $D \in C_2(S^3)$. Note that D was not available in homology. Now we can consider $m_1(D) \in C_3(S^3)$; this corresponds to filling up the 3-sphere with a solid torus, and this is nontrivial. This gives us a secondary operation $m_3 : H_0(S^3) \to H_3(S^3)$. (In general operations like m_3 are only defined on the kernel of operations like m_1.) Operations like this can also fall out of looking at spectral sequences.

4.2 Cones

In the theory of triangulated categories there is some extra structure that needs to be attached to talk about triangles and cones. In the dg-category picture we can do this more directly. There are two dg-categories $C^{\mathrm{alg}}_{\mathrm{cone}}$ and $C^{\mathrm{top}}_{\mathrm{cone}}$ which are equivalent and both of which universally describe cones. These categories should have three objects x, y, z with maps $f : x \to y, g : y \to z, h : z \to x$ where h has degree 1, and all non-identity compositions are zero in homology. Starting from this data, we should freely add morphisms $F : y \to x, G : z \to y, H : x \to z$ where H has degree -1. In addition, we impose relations

$$df = 0, dg = 0, dh = 0 \tag{62}$$

and

$$dH = g \circ f, dG = f \circ h, dF = h \circ g. \tag{63}$$

This situation has some nontrivial Massey products which we will set to zero by setting

$$\mathrm{id}_x = F \circ f - h \circ H, \mathrm{id}_Y = G \circ g - f \circ F, \mathrm{id}_z = H \circ h - g \circ G. \tag{64}$$

(If we hadn't done this, then after passing to homology we would get extra endomorphisms of x, y, z.) This defines $C^{\mathrm{alg}}_{\mathrm{cone}}$. Roughly speaking what we did above was identify two different reasons why compositions like $h \circ g \circ f$ are zero after passing to homology.

Exercise 4.3. *A dg-functor from C^{alg}_{cone} to a dg-category C is the same thing as a cone in C.*

$C^{\text{top}}_{\text{cone}}$ is a full dg-subcategory of the category of sheaves of chain complexes on $\mathbb{R}$. x will be the sheaf of cochains on $[0,\infty)$, y will be the sheaf of cochains on $(-\infty,\infty)$, and z will be the sheaf of cochains on $(-\infty,0)$.

4.3 Localization

Let C be a dg-category and N a full subcategory. We want to construct a quotient dg-category C/N with the universal property that every object in N is sent to the zero object.

Drinfeld's construction of the quotient D has the same objects as C. The morphisms are the same as in C except that we freely add morphisms ε_x of every object $x \in N$ of degree -1 such that $d\varepsilon_x = \text{id}_x$.

Example Let C have one object x with endomorphisms $\mathbb{C}\text{id}_X$ and let $N = C$. Then C/N has one object x with endomorphisms $\mathbb{C}\varepsilon_x^n$ in degree $-n$. Furthermore, $d(\varepsilon^n) = 0$ for all $n \geq 2$, so the cohomology of $\text{End}(x)$ is trivial and C is (weakly) equivalent to the zero category.

Definition Let A be a dg-algebra. In the module category Mod-A of A, consider the subcategory N of acyclic modules. The *unbounded derived category* $D(A)$ is the quotient of Mod-A by N.

Exercise 4.4. *The natural inclusions of both complexes of projective modules and complexes of injective modules into $D(A)$ are equivalences.*

Remark Let C be the one-object dg-category associated to A. The stabilization C^{st} sits as a fully faithful subcategory of $D(A)$. Its image is sometimes called the category $\text{Perf}(A)$ of perfect complexes over A.

Lemma 4.5. *Quotienting by acyclic modules is equivalent to inverting quasi-isomorphisms.*

Proof. One direction is clear (consider the unique map from an acyclic complex to the zero complex). In the other direction, suppose $f : X \to Y$ is a quasi-isomorphism. Then cone(f) is acyclic. If it is killed via an endomorphism $\varepsilon_{\text{cone}(f)}$ of degree -1, then the composition

$$Y \to \text{cone}(f) \xrightarrow{\varepsilon} \text{cone}(f) \to X \tag{65}$$

(where the middle morphism has degree -1 but the right morphism has degree $+1$) is an inverse to f in the derived category. □

4.4 Application to algebraic geometry

Let X be a complex variety.

Definition The *derived category of quasicoherent sheaves* $D_{\text{QCoh}}(X)$ on X is the quotient of the dg-category of complexes of quasicoherent sheaves by the acyclic sheaves. Similarly, $D_{\text{Coh}}(X)$ is the quotient of hte dg-category of complexes of quasicoherent sheaves with (bounded) coherent cohomology sheaves by the acyclic sheaves.

Exercise 4.6. *$D_{Coh}(X)$ is equivalent to the quotient of the dg-category of bounded complexes of coherent sheaves by the acyclic sheaves.*

The original motivation for this definition is that if $f : X \to Y$ is a nice map, we want to think about pushforward and pullback of quasicoherent sheaves, but these are not exact functors. Passing to the left and right derived functors, which are naturally defined on the derived categories, fixes this.

The new motivation for this definition is that $D_{\text{Coh}}(X)$ is easier to deal with than Coh(X). If X is affine, then Coh(X) is the category of finitely-generated modules over $\mathcal{O}(X)$, so is straightforward to deal with. If X is projective, then $D_{\text{Coh}}(X)$ is also straightforward to deal with in this sense.

Example Let $X = \mathbb{P}^1$. The category Coh$(\mathbb{P}^1)$ of coherent sheaves is complicated; it consists of direct sums of vector bundles adn torsion sheaves (e.g. Jordan sheaves). The derived category is substantially simpler; it is the category of perfect complexes over the Kronecker quiver, or equivalently over the triangular ring

$$A = \begin{bmatrix} \mathbb{C} & 0 \\ \mathbb{C}^2 & \mathbb{C} \end{bmatrix}. \tag{66}$$

The derived category of coherent sheaves is a strong invariant and can be calculated; today we will calculate it for $\mathbb{P}^n$. The ordinary category of coherent sheaves can be recovered with additional structure (a t-structure), but many interesting equivalences relating to other mathematics do not respect this structure.

What does it mean to calculate a derived category C? Our goal will be to find a dg-algebra A such that C is equivalent to $\mathrm{Perf}(A)$. (A is not unique.)

Definition Let $C \cong C^{st}$ be a (pre)triangulated or stable category. An object $c \in C$ is a (strong, triangulated) *generator* if the natural inclusion from the stabilization of the one-object full subcategory on c is an equivalence.

Proposition 4.7. *Let C be stable and let $c \in C$ be a strong generator. Let $A = Hom(c, c)$. Then the Yoneda map*

$$C \ni d \mapsto Hom(c, d) \in Perf(A) \tag{67}$$

is an equivalence.

Proof. Resolve d by shifts and summands of c. □

Exercise 4.8. *Find an A with the above property for an elliptic curve.*

Proposition 4.9. $c = \mathcal{O}(-1) \oplus \mathcal{O} \in D_{Coh}(\mathbb{P}^1)$ *is a strong generator.*

The dg-algebra we want is therefore $A = \mathrm{End}(c)$. This is the triangular ring

$$\begin{bmatrix} \mathbb{C} & 0 \\ \mathbb{C}^2 & \mathbb{C} \end{bmatrix} \tag{68}$$

because there are no morphisms $\mathcal{O} \to \mathcal{O}(-1)$ in the derived category but there are two linearly independent morphisms $\mathcal{O}(-1) \to \mathcal{O}$. This is the path algebra over the Kronecker quiver $\bullet \rightrightarrows \bullet$.

Proof. Recall that $\mathrm{Coh}(\mathbb{P}^1)$ consists of direct sums of vector bundles, which we understand by Grothendieck's theorem, and torsion sheaves (direct sums of Jordan sheaves). It suffices to show that all coherent sheaves are complexes of $\mathcal{O} \oplus \mathcal{O}(-1)$ and their shifts. (Given a complex with bounded coherent cohomology, we can truncate it, which gives a quasi-isomorphic complex that is bounded below. We can then start resolving its cohomology one piece at a time.) First, we have a resolution

$$\mathcal{O}(-1) \xrightarrow{f} \mathcal{O} \to \mathbb{C}_{\mathrm{pt}} \tag{69}$$

giving us the structure sheaf of a point. Any torsion sheaf is obtained by a sequence of extensions of such structure sheaves, so we can obtain any torsion sheaf.

It now suffices to show that we can obtain any line bundle. Tensoring the above resolution by $\mathcal{O}(k)$ gives

$$\mathcal{O}(k-1) \to \mathcal{O}(k) \to \mathbb{C}_{\mathrm{pt}}. \tag{70}$$

A pair of inductions gives us every line bundle as desired. □

Theorem 4.10. *$c = \mathcal{O} \oplus \mathcal{O}(-1) \oplus ... \oplus \mathcal{O}(-n) \in D_{Coh}(\mathbb{P}^n)$ is a strong generator.*

$A = \mathrm{End}(c)$ is again a triangular ring. It is the path algebra of an acyclic quiver with $n+1$ vertices.

Proof. We will give three versions of the proof.

Version 1: this story is local and takes place on $\mathbb{A}^n$ (Koszul resolution). Suppose we want to resolve $\mathbb{C}_x$ by a complex of vector bundles. There is a surjection $\mathcal{O}_{\mathbb{A}^n} \to \mathbb{C}_x$. The kernel can be generated by n elements, but writing down a surjection from n copies of $\mathcal{O}_{\mathbb{A}^n}$ is not very invariant. A better way to write this is $\mathcal{O}_{\mathbb{A}^n} \otimes \Omega^1_x$ (where Ω^1_x is 1-forms at x). This extends to a resolution

$$\mathbb{C}_x \leftarrow \mathcal{O}_{\mathbb{A}^n} \leftarrow \mathcal{O}_{\mathbb{A}^n} \otimes \Omega^1_x \leftarrow ... \leftarrow \mathcal{O}_{\mathbb{A}^n} \otimes \Omega^n_x. \tag{71}$$

(The fact that this resolution is finite is equivalent to the statement that 0 is smooth. This is the Auslander-Buchsbaum theorem.)

Version 2: we want to resolve a skyscraper $\mathbb{C}_x$ on $\mathbb{P}^n$. Our first step is to write down an epimorphism $\mathcal{O}_{\mathbb{P}^n} \to \mathbb{C}_x$. The next step is to write down an epimorphism from $\mathcal{O}_{\mathbb{P}^n}(-1) \otimes \Omega^1_x$, and similar to the above we get a resolution

$$\mathbb{C}_x \leftarrow \mathcal{O}_{\mathbb{P}^n} \leftarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \Omega^1_x \leftarrow ... \leftarrow \mathcal{O}_{\mathbb{P}^n}(-n) \otimes \Omega^n_x. \tag{72}$$

This is not the end of the story because there are more objects running around than torsion sheaves and vector bundles. □

4.5 Integral transforms

We can think of sheaves as categorified functions. There are many interesting operations in analysis available on functions, e.g. pullback, multiplication, integration... putting these operations together, we can define integral transforms

$$\Phi_K(f)(y) = \int_X f(x)K(x,y)\,x \tag{73}$$

where $K(x,y)$ is a function on a product space $X \times Y$, f is a function on X, and the integral transform is a function on y. The Fourier transform arises in this way. We will categorify this formalism.

Let X, Y be nice varieties and consider the diagram

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y. \tag{74}$$

Given a kernel $K \in D_{\text{QCoh}}(X \times Y)$, we get an *integral transform* functor

$$D_{\text{QCoh}}(X) \ni M \mapsto \Phi_K(M) = R(p_Y)_* \left(Lp_X^*(M) \otimes^L K \right) \in D_{\text{QCoh}}(Y) \tag{75}$$

(so derived pullback, derived tensor product, then derived pushforward, which categorifies the analytic picture). We can think of Φ as a functor in K, hence a functor

$$\Phi : D_{\text{QCoh}}(X \times Y) \to \text{Funct}^{\text{cont}} \left(D_{\text{QCoh}}(X), D_{\text{QCoh}}(Y) \right) \tag{76}$$

where the RHS is continuous (direct-sum-preserving) dg-functors. This is an equivalence. In other words, reasonable functors between derived categories can be described using integral kernels. If X, Y are smooth and proper, we can replace quasi-coherent with coherent everywhere above. Hence to write down derived equivalences it suffices to write down suitable kernels.

Example Let $f : X \to Y$ be a map. To it we can associate functors Lf^* and Rf_* which ought to be representable by integral kernels. These turn out to both correspond to the structure sheaf of the graph Γ_f of f in $X \times Y$ (alternately, $\Gamma_f : X \to X \times Y$ is a morphism and we want to take the derived pushforward of the structure sheaf along this map).

Example Given $F \in D_{\text{Coh}}(X)$, we get a functor $- \otimes^L F : D_{\text{Coh}}(X) \to D_{\text{Coh}}(X)$. The corresponding integral kernel is the (derived?) pushforward of F along the diagonal embedding $\Delta : X \to X \times X$.

Example Let $K = K_1 \boxtimes K_2$ where K_1 is a kernel on X and K_2 is a kernel on Y. Then it turns out that

$$\Phi_K(M) = K_2 \otimes^L \left(R\Gamma(K_1 \otimes^L M) \right). \tag{77}$$

This is a kind of projection.

We return to the third version of the proof above, which is due to Beilinson; the resolution we will construct is called the Beilinson resolution. We have a resolution of the skyscraper sheaves $\mathbb{C}_x$. We would now like to vary x across the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. This globalizes the above resolution to a resolution

$$\mathcal{O}_\Delta \leftarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \leftarrow \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1_{\mathbb{P}^n} \leftarrow ... \leftarrow \mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes \Omega^n_{\mathbb{P}^n} \tag{78}$$

(where by $F_1 \boxtimes F_2$ we mean $(Lp_1^* F_1) \otimes^L (Lp_2^* F_2)$).

Let X, Y be nice varieties. Given a quasicoherent sheaf $K \in D_{\text{QCoh}}(X \times Y)$, we get an integral transform as above. If $K = \mathcal{O}_\Delta$, then Φ_K is the identity functor. But we now have an interesting resolution of $\mathcal{O}_\Delta$, which gives us an interesting resolution of the identity functor. We conclude that

$$M \cong \Phi_{\mathcal{O}_\Delta}(M) \cong \Phi_{B^\bullet}(M) \tag{79}$$

where $B^\bullet$ is the resolution above. Applying $\Phi_{B^\bullet}$ and keeping the above example about integral kernels of the form $F_1 \boxtimes F_2$ in mind, we conclude that M lies in the stabilization of the full subcategory on $\mathcal{O}, ...\mathcal{O}(-n)$ as desired.

Let V be a finite-dimensional real vector space equipped with a volume form. The Fourier transform furnishes an isomorphism between functions on V and functions on V^* given by

$$\hat{f}(\lambda) = \int_V f(v) e^{i\lambda(v)} \, dv. \tag{80}$$

The Fourier transform is nontrivial; it sends delta functions to exponentials and vice versa.

The Fourier-Mukai transform generalizes this to derived categories of abelian varieties: for an elliptic curve E, it furnishes an integral transform

$$\Phi_P : D_{\mathrm{Coh}}(E) \to D_{\mathrm{Coh}}(E^\vee) \tag{81}$$

where $E^\vee$ is the dual abelian variety; here it can be thought of as $\mathrm{Pic}^0(E)$, and it is isomorphic to E. The integral kernel $P \in D_{\mathrm{Coh}}(E \times E^\vee)$ will be the universal (tautological, Poincaré) line bundle defined as follows: a point in $E^\vee$ is a line bundle L, and the restriction of P to points with coordinate L is L.

Theorem 4.11. *Φ_P is an equivalence. More precisely,*

$$\Phi_P \circ \Phi_{P^\vee} \cong (-1)^*[-\dim E]. \tag{82}$$

(where -1 is the negation map $E \to E$ and $[-\dim E]$ is a shift.) Φ_P takes convolution to tensor product and takes tensor product to convolution shifted by 1. (Convolution is given by taking the external product of two sheaves and pushing it forward under the addition map.)

Proof. We want to compute a composition of integral transforms $X \to Y \to Z$. Starting with a kernel K_{XY} on $X \times Y$ and a kernel K_{YZ}, the kernel describing the composite is given by pulling back K_{XY} and K_{YZ} to $X \times Y \times Z$, tensoring them, and pushing it forward to $X \times Z$. In symbols,

$$K_{XY} \circ K_{YZ} = Rp_*(Lf^* K_{XY} \otimes^L Lg^* K_{YZ}) \tag{83}$$

where $f : X \times Y \times Z \to X \times Y$ and $g : X \times Y \times Z \to Y \times Z$ are the projections.

We want to compute $F = P \circ P^\vee$. This is an integral kernel on $E^\vee \times E^\vee$ which is the pushforward of some $\tilde{F}$ on $E^\vee \times E \times E^\vee$. The sheaf $\tilde{F}$ is the line bundle whose restrictions to the subspace with coordinates $(L_1, -, L_2)$ is the tensor product $L_1 \otimes L_2$.

The claim now is that $F = Rp_*(\tilde{F})$ is $\mathcal{O}_{\Gamma_{-1}}[-1]$ (the structure sheaf of the graph of the -1 map, shifted by -1). Roughly speaking this is because a generic line bundle on E of degree 0 has no global sections; indeed, by Riemann-Roch, the Euler characteristic of a line bundle of degree 0 is $1 - 1 + 0 = 0$, hence H^0 is nontrivial iff H^1 is nontrivial. Hence $L_1 \otimes L_2$ has global sections precisely when $L_2 \cong L_1^*$, and taking duals of line bundles is the -1 map on $E^\vee$.

So far we've argued that F must be supported on the graph Γ_{-1}. We can think of F as a family of chain complexes over $E^\vee \times E^\vee$. Equivalently, we can think of F as a chain complex of vector bundle terms with differential varying on $E^\vee \times E^\vee$; in fact, it has the form $C^0 \xrightarrow{d(L_1,L_2)} C^1$. The kernel of this map is a vector bundle, hence is non-empty iff it is non-empty generically. Since it vanishes generically, it vanishes on Γ_{-1}.

(To be continued...) □

4.6 Fourier-Mukai and flops

Today's theme will be base change for varieties. Consider a diagram of the form

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{84}$$

and let $F \in D_{\mathrm{QCoh}}(X)$. Assume that g is *flat*, so that $\mathcal{O}_{Y',y'}$ is flat over $\mathcal{O}_{Y,y}$ whenever $g(y') = y$. The upshot of this is that $Lg^* = g^*$.

Theorem 4.12. *With setup as above, there is a natural isomorphism of functors*

$$g^* Rf_* \cong Rf'_* g' : D_{QCoh}(X) \to D_{QCoh}(Y'). \tag{85}$$

Proof. We work locally on Y, Y'. Assume $Y = \mathrm{Spec}\ A, Y' = \mathrm{Spec}\ A'$. Then one of these functors is $R\Gamma(X, F) \otimes_A A'$ and the other is

$$R\Gamma(X \times_Y Y', g^* F) \cong R\Gamma(X \times_Y Y', F \otimes_A A'). \tag{86}$$

Flatness implies that $R\Gamma$ commutes with $\otimes_A A'$ and the conclusion follows.

Alternately, we may assume that F is flat over Y in the sense that F_x is flat over $\mathcal{O}_{Y,y}$ whenever $f(x) = y$. □

Earlier we reduced the proof that the Fourier-Mukai transform is an equivalence to the claim that $Rp_*(\tilde{F}) \cong \mathcal{O}_{\Gamma_{-1}}[-1]$. Now, we have

$$Rp_*(\tilde{F})|_{(x,y)} \cong R\Gamma(E, L_x \otimes L_y). \tag{87}$$

If i denotes the inclusion of (x, y) into $E^\vee \times E^\vee$, then by base change we have

$$Li^* Rp_*(\tilde{F}0 \cong R\Gamma(E, L_x \otimes L_y) \tag{88}$$

which vanishes unless $L_x \cong L_y^{-1}$, from which it follows that the stalks of $Rp_*(\tilde{F})$ vanish off of Γ_{-1}. On Γ_{-1} we must have $Rp_*(\tilde{F}) \cong \mathcal{O}_{\Gamma_{-1}}[-1]$ by a straightforward computation.

(Something about flop equivalences.)

We also have the following reconstruction theorem: suppose X is a smooth projective variety such that ω_X is ample. Then if $D_{\text{Coh}}(X) \cong D_{\text{Coh}}(Y)$, then $X \cong Y$.

4.7 Serre duality

Let X be a smooth proper variety of dimension n with canonical bundle Ω_X^n. Let $\omega_X = \Omega^n X[n]$.

Theorem 4.13. *(Serre duality) Let $F \in D_{Coh}(X)$. Then*

$$H^i(X; F) \cong H^{n-i}(X; F^* \cong \Omega_X^n)^*. \tag{89}$$

For example, setting $X = \mathbb{P}^1$ and $F = \mathcal{O}(n)$ this explains the relationship we saw between H^0 and H^1 of line bundles. Serre duality can also be stated

$$H^i(X, F) \cong H^{-i}(X; F^* \otimes \omega_X)^*. \tag{90}$$

A special case, roughly speaking, is the isomorphism $\text{Hom}(V, W) \cong V^* \otimes W$ for V, W finite-dimensional vector spaces. In fact, Serre duality can also be stated in a form like this. First, define

$$\text{Hom}_{D_{\text{Coh}(X)}}(V, W) \cong R\Gamma(X; \text{Hom}(V, W)). \tag{91}$$

Then Serre duality implies and is equivalent to the claim that

$$\text{Hom}_{D_{\text{Coh}}(X)}(V, W) \cong \text{Hom}_{D_{\text{Coh}}(X)}(W, V \otimes \omega_X)^*. \tag{92}$$

This suggests the following definition.

Definition Let C be a (Hom-finite) dg-category. A *Serre functor* is a functor $S : C \to C$ such that there is a natural isomorphism

$$\text{Hom}(x, y) \cong \text{Hom}(y, S(x))^*. \tag{93}$$

So Serre duality is the claim that $- \otimes \omega_X$ is a Serre functor.

Let $f : X \to \text{pt}$ be the unique map. Then

$$\text{Hom}_{D_{\text{Coh}}(\text{pt})}(Rf_* F, V) \cong (Rf_* F)^* \otimes V \cong Rf_*(F \otimes W) \otimes V \tag{94}$$

by Serre duality. This is in turn just $\text{Hom}_{D_{\text{Coh}}(X)}(F, \omega_X \otimes V)$. We can write $\omega_X \otimes -$ as a functor $f^!$, and then this is an adjunction.

Theorem 4.14. *(Grothendieck-Serre duality) There is an adjunction $(f_*, f^!)$ where $f : X \to Y$ is smooth and proper. More specifically,*

$$f^! : M \mapsto (Lf^* M) \otimes \omega_{X/Y} \tag{95}$$

where $\omega_{X/Y}$ is the relative canonical bundle.

4.8 The final project

The final project will be a research proposal. It should begin with some background on something you're interested in (e.g. Serre duality). It should continue with a question or a conjecture. At the end, it should take some first steps (e.g. calculations). Alternatively, the final project would be a ten-minute presentation. This should happen on May 9th or so.

4.9 Analogy to Poincaré duality

Serre duality is analogous to Poincaré duality. The analogy proceeds as follows:

Smooth projective variety	Smooth compact manifold
The structure sheaf $\mathcal{O}_X$	The sheaf of locally constant functions $\mathbb{C}_M$
The derived category of coherent sheaves	The derived category of local systems

Let X be a nice space. The fundamental groupoid $\Pi_1(X)$ has objects the elements of X and morphisms the homotopy classes of paths between points of X. Isomorphism classes correspond to path components. A *local system* on X is a functor $\Pi_1(X) \to \text{FinVect}$. A local system may be presented as a locally constant sheaf, and then by the derived category of local systems we mean the derived category of $\mathbb{C}_M$-modules whose cohomologies are bounded and locally constant sheaves.

Example Let $M = S^1$. Then a local system on M is precisely a vector space equipped with an automorphism. The derived category here is the obvious thing, namely every object is isomorphic to a complex of local systems.

Example Let $M = S^2$ and let $h : S^3 \to S^2$ be the Hopf fibration. Then the derived pushforward $Rh_* \mathbb{C}_{S^3}$ is not a complex of local systems.

The cohomology $H^\bullet(M; \mathcal{L})$ of a manifold with values in a local system is the derived pushforward of $\mathcal{L}$ to a point, as in the algebraic setting. There is also a more classical picture. Take some topological space. We can think about 0-chains as dots with numbers attached to them and 1-chains as oriented line segments with numbers attached to them, and so forth. Instead of assigning numbers, we can think about assigning elements of a local system.

Poincaré duality says that

$$H^i(M; \mathcal{L})^* \cong H^{n-i}(M, \mathcal{L}^* \otimes \text{or}_M) \tag{96}$$

where or_M is the *orientation local system* assigning to every point in the manifold the (one-dimensional) vector space of orientations at that point. Equivalently, there is a dualizing complex $\omega_M = \text{or}_M[n]$.

Poincaré duality is usually stated in terms of homology; here is the relationship. Recall that cochains $C^\bullet(Z)$ is the dual of chains $C_{-\bullet}(Z)$. In fact there is a cap product

$$C^a(Z) \otimes C_{-b}(Z) \to C_{-b-a}(Z) \tag{97}$$

and pushing forward $C_{-b-a}(Z) \to C_{-b-a}(\text{pt})$ gives the dual pairing. The local picture pairs sheaves of cochains and cosheaves of chains to get cosheaves of chains. We like sheaves, so instead of cosheaves we can work with sheaves of Borel-Moore chains. Borel-Moore chains on Y are the relative chains on the one-point compactification of Y relative to infinity.

If Z is compact, then Borel-Moore chains are just chains. (This is interesting because we are in some sense identifying a limit and a colimit.) If Z is smooth, then Borel-Moore chains can be identified with $\text{or}_Z[\dim Z]$. This observation is in some sense Poincaré duality.

4.10 Dualizing complexes

Previously we showed that the (derived?) pushforward $f_* : D_{\mathrm{QCoh}}(X) \to D_{\mathrm{QCoh}}(Y)$ admits a right adjoint $f^!$. If X is a variety and Y a point, then $\omega_X = f^! \mathbb{C}_{\mathrm{pt}}$ is a dualizing complex for X (in that it manifests Serre duality for X).

Example If $X = \mathbb{P}^n$, then $\omega_X = \Omega^n_{\mathbb{P}^n}[n] \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)[n]$.

Theorem 4.15. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Then $\omega_X \cong \mathcal{O}_X(-n-1+d)[n-1]$.*

If X is smooth, we have $\omega_X \cong \Omega_X^{\dim X}[\dim X]$. If X is not smooth, then ω_X will be more complicated. Roughly speaking we have the following set of analogies.

Smooth proper variety, $\omega_X \cong \Omega_X^{\dim X}[\dim X]$	Smooth compact manifold
Gorenstein, ω_X is a line bundle shifted by $\dim X$	Topological manifold
Cohen-Macaulay, ω_X is a sheaf shifted by $\dim X$	Locally a cone over a wedge of spheres
General case, ω_X is a complex	General case

Research problem: homology is right adjoint to cohomology in this class. ω_X independent of adjunction?

Example Let $R = \mathbb{C}[x]/x^2$ and let $X = \operatorname{Spec} R$. X is Gorenstein. There is an adjunction $(f_*, f^!)$ between R-modules and $\mathbb{C}$-modules defined by taking

$$f^!(V) = \mathrm{Hom}_{\mathbb{C}}(R, V) \cong R^* \otimes_{\mathbb{C}} V \cong R \otimes_{\mathbb{C}} V. \tag{98}$$

So in this case the dualizing complex is $\omega_X \cong R$.

Example Let X be a cuspidal $\mathbb{P}^1$. Functions on X can be identified with the ring $\mathbb{C}[t^2, t^3]$. Check that

$$H^\bullet(X, \mathcal{O}_X(n)) \cong H^{1-\bullet}(X, \mathcal{O}_X(-n) \otimes_{\mathcal{O}_X} \omega_X)^* \tag{99}$$

where $\omega_X \cong \mathcal{O}_X[1]$.

5 Reconstruction theorems

Theorem 5.1. *(Bondal-Orlov) Let X be a smooth irreducible projective variety with either ample or anti-ample canonical bundle. If $D_{Coh}(X) \cong D_{Coh}(X')$ then $X \cong X'$.*

To prove this theorem we will need the following.

Definition Let C be a dg-category with a Serre functor $S : C \to C$. An object $P \in C$ is a *point object* of codimension s if the following conditions hold:

1. $SP \cong P[s]$.
2. $\mathrm{Hom}^{<0}(P, P) \cong 0$.
3. $\mathrm{Hom}^0(P, P) \cong \mathbb{C}$.

The structure sheaf of a point is a point-object of codimension $\dim X$.

Proposition 5.2. *Let $C = D_{Coh}(X)$ and let X be a smooth irreducible projective variety with Ω_X^n ample or anti-ample. then $P \in D_{Coh}(X)$ is a point object iff it has the form $\mathcal{O}_x[k]$ for $x \in X$ (in particular it has codimension n).*

Definition Let C be a dg-category. $L \in C$ is *invertible* if for any point object $P \in C$ there exists $s \in \mathbb{Z}$ such that

1. $\mathrm{Hom}^s(L, P) \cong \mathbb{C}$
2. $\mathrm{Hom}^i(L, P) \cong 0$ for $i \neq s$.

Proposition 5.3. *Let $C = D_{Coh}(X)$ and assume that point objects have the form $\mathcal{O}_x[s]$. Then $L \in C$ is invertible iff it is a shift of a line bundle.*

Proof. One direction is an exercise. In the other direction, look at the top cohomology sheaf $H^{\mathrm{top}}(L)$. Then for x in the support of $H^{\mathrm{top}}(L)$ we have

$$\mathrm{Hom}^0(H^{\mathrm{top}}(L), \mathcal{O}_x) \cong \mathbb{C} \tag{100}$$

and

$$\mathrm{Hom}^1(H^{\mathrm{top}}(L), \mathcal{O}_x) = 0. \tag{101}$$

This follows from the invertibility of L. It follows that $H^{\mathrm{top}}(L)$ is locally free and rank 1. Similarly we see that $H^{\mathrm{top}-i}(L) = 0$ for $i > 0$. $\square$

We now prove Bondal-Orlov. With $C = D_{\text{Coh}}(X), C' = D_{\text{Coh}}(X')$, let $P_C, P_{C'}, P_X, P_{X'}$ denote the sets of point objects in C, C', X, X' respectively. We know that $P_X \subseteq P_C$ and $P_{X'} \subseteq P_{C'}$. We also know that $P_C \cong P_{C'}$.

Step 1: We show that $P_{X'} = P_{C'}$. Suppose P is a point object in P_C not in $P_{X'}$. Then $P = \mathcal{O}_x[k]$ for some $x \in X$, but there are no nonzero homomorphisms between P and $\mathcal{O}_{x'}$ for any $x' \in X'$. But then $P \cong 0$ since it has no stalks. We conclude that X, X' have the same points up to shift.

Step 2: Applying the proposition, we conclude that X, X' have the same line bundles up to shift.

Step 3: Fix $L_0 \in C$ which is invertible, shifted so that it is a line bundle on X'. Consider

$$p_C = \{p \in P_C : \text{Hom}(L_0, P) = \mathbb{C}\} \tag{102}$$

(without a shift). This can be identified with both the points of X (unshifted) and the points of X' (unshifted).

Step 4: We want to produce the Zariski topology. Define

$$\ell_C = \ell_X = \ell_{X'} = \{L \in L_C : \text{Hom}(L, P) = \mathbb{C} \forall p \in p_C\}. \tag{103}$$

For $L_1, L_2 \in \ell_C$ take $\alpha \in \text{Hom}(L_1, L_2)$. It induces a map $\alpha_P^* : \text{Hom}(L_2, P) \to \text{Hom}(L_1, P)$. The Zariski topology can be recovered by looking at those P such that $\alpha_P^* \neq 0$.

Step 5: We want to find an embedding into projective space. Construct the graded algebras

$$A = \bigoplus_i \text{Hom}_C(L_0, L_0 \otimes (\Omega_X^n)^i) \tag{104}$$

$$A' = \bigoplus_i \text{Hom}_{C'}(L_0, L_0 \otimes L^i) \tag{105}$$

where L is the line bundle such that $-\otimes L$ is the Serre functor. The Proj of these graded algebras should give us the original varieties. We need to know that $\Omega^n_{X'}$ is also ample.

6 Borel-Weil-Bott

The irreducible (finite-dimensional) representations of $\mathrm{SL}_2(\mathbb{C})$ are in bijection with the non-negative integers. They are given by $V_n = S^n(\mathbb{C}^2)$, or equivalently the space of homogeneous polynomials of degree n in two variables.

Exercise 6.1. *Compute* $V_n \otimes V_m$*.*

Exercise 6.2. *The representations* V_n *are all self-dual. Describe nondegenerate pairings* $V_n \times V_n \to \mathbb{C}$*.*

The Borel-Weil-Bott theorem, among other things, lets us write down irreducible representations systematically for reductive affine algebraic groups. Borel-Weil tells us the following. Given a non-negative integer n we can associate to it the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. These are in fact equivariant line bundles.

Definition Let X be a variety and let G be an affine algebraic group acting on X. A sheaf F is *equivariant* if the pullback of F along either the projection or action maps $G \times X \to X$ are isomorphic and moreover, if α denotes this isomorphism, it should be compatible with the commutative diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times p} & G \times X \\ \big\downarrow{\scriptstyle \mathrm{id}_G \times a} & & \big\downarrow{\scriptstyle a} \\ G \times X & \xrightarrow{a} & X \end{array} \qquad (106)$$

Exercise 6.3. *There exists a unique* $SL_2(\mathbb{C})$*-equivariant structure on* $\mathcal{O}(n)$*.*

It follows that $\mathrm{SL}_2(\mathbb{C})$ acts on global sections $\Gamma(\mathbb{P}^1, \mathcal{O}(n))$. The Borel-Weil theorem in this case says that this representation is V_n. Borel-Weil-Bott tells you that V_n is also $H^1(\mathbb{P}^1, \mathcal{O}(-n-2))$.

The generalization to $\mathrm{SL}_{n+1}(\mathbb{C})$ is as follows. We should think of representations as being parameterized by $\mathbb{N}^n \cong \mathbb{Z}^n / S_{n+1}$ where the action of S_{n+1} comes from the action of $\mathbb{Z}^{n+1}$ (but we take the subgroup of vectors summing to zero). We will assign to the vector in $\mathbb{N}^n$ with i^{th} coordinate 1 and other coordinates 0 the i^{th} exterior power $\Lambda^i(\mathbb{C}^{n+1})$ of the defining representation.

Let B be the Borel subgroup of upper triangular matrices in $\mathrm{SL}_{n+1}(\mathbb{C})$. The quotient $X = \mathrm{SL}_{n+1}(\mathbb{C})/B$ is called a *flag variety*. It parameterizes complete flags (nested sequences of subspaces) in $\mathbb{C}^{n+1}$. This is a (smooth projective) variety because it embeds into a product of Grassmannians and is cut out by certain equations, the Plücker relations.

Exercise 6.4. *SL_{n+1}-equivariant line bundles on X are naturally in bijection with $\mathbb{Z}^n$.*

If $\lambda \in \mathbb{Z}^n$ then we will denote the corresponding line bundle by $\mathcal{O}(\lambda)$. Borel-Weil says that $\mathrm{SL}_{n+1}(\mathbb{C})$ acts on $H^0(X; \mathcal{O}(\lambda))$ and that for λ dominant this is an irreducible representation V_λ. Bott says that $\mathcal{O}(\lambda)$ either has trivial cohomology or cohomology nontrivial in exactly one degree which gives an irreducible representation.

In general, the Bott picture suggests that we recenter our picture at the square root of the canonical bundle. Drawing hyperplanes through this bundle we get various Weyl chambers. Line bundles on the walls of these chambers have no cohomology; otherwise, the nontrivial cohomology occurs in H^i where i is the number of walls necessary to cross to get to the chamber (and is an irreducible representation).

Let G be an algebraic group and let X be a variety on which G acts transitively.

Lemma 6.5. *The category of equivariant coherent sheaves on X is precisely the category of finite-dimensional representations of a stabilizer subgroup of a point of X.*

In particular, G-equivariant line bundles are the same as characters of a stabilizer.

Let $G = \mathrm{SL}_n(\mathbb{C})$ and let B be the Borel subgroup. Then G-equivariant line bundles on the flag variety G/B are the same as characters of B. And characters of B are easy to understand; any such thing factors through B/U where U is the unipotent subgroup. We will call this group H. The lattice we have been looking at is the lattice of characters of H. The trivial bundle corresponds to the trivial character. Now we need to find the canonical bundle.

First, the tangent bundle, as a representation of B, corresponds to $\mathfrak{g}/\mathfrak{b}$ via the adjoint action. The cotangent bundle corresponds to the dual $(\mathfrak{g}/\mathfrak{b})^*$, so the canonical bundle corresponds to $\Lambda^n((\mathfrak{g}/\mathfrak{b})^*)$.

We can identify $(\mathfrak{g}/\mathfrak{b})^*$ with $\mathfrak{u}$ via an invariant inner product; for $\mathrm{SL}_n(\mathbb{C})$ we can take $\mathrm{tr}(ab)$. When $n = 3$ this representation will be a sum of three characters. Their top exterior power is their tensor product, and this is the canonical bundle.

When $G = \mathrm{SL}_n(\mathbb{C})$, and more generally, there is a Weyl group action on the lattice of characters and we want to recenter it at the square root of the canonical bundle. The Weyl group is generated by so-called simple reflections, which correspond to simple roots and to simple $\mathbb{P}^1$ fibrations. These are defined as follows when $G = \mathrm{SL}_n(\mathbb{C})$. If X is the complete flag variety, for each i there is a map $\pi_i : X \to X_i$ where X_i is a partial flag variety in which we have ignored the i^{th} subspace in a flag. These are all $\mathbb{P}^1$ fibrations. Geometrically we might therefore hope to reduce the general situation to the case of $\mathbb{P}^1$.

The relative canonical bundles of the π_i turn out to correspond to the simple roots. We want to apply relative Serre duality. It will turn out that if we take two line bundles related by a simple reflection, then they are relatively Serre dual along π_i. Furthermore, we can push them forward along π_i and the pushforwards will be isomorphic up to a shift on X_i. This allows us to reduce Borel-Weil-Bott to the case of dominant weights.

Remark Where does Borel-Weil-Bott come from, if you're a representation theorist? In representation theory we want to construct representations. Usually only the trivial representation suggests itself. The regular representation is usually too big. The only other method is induction. Borel-Weil-Bott can be seen in this framework.

First, $\mathrm{Coh}^G(\mathrm{pt})$ (G-equivariant coherent sheaves on a point) can be identified with G-representations. Induction from K-representations to G-representations (K a subgroup of G) is some functor $\mathrm{Coh}^K(\mathrm{pt}) \to \mathrm{Coh}^G(\mathrm{pt})$. Now, K-equivariant sheaves can be identified with G-equivariant sheaves on G/K, and now induction should be a functor $\mathrm{Coh}^G(G/K) \to \mathrm{Coh}^G(\mathrm{pt})$, and it should be adjoint to restriction. This corresponds to pushforward (induction) and pullback (restriction) along the unique map $G/K \to \mathrm{pt}$.

But we don't have a pushforward of coherent sheaves unless G/K is proper. It's natural to try to induct from a maximal torus T, but G/T is not proper. G/B is a version of G/T which is proper, and B has the same characters as T. So looking at equivariant coherent sheaves on the flag variety G/B is very natural.