253 Homological Algebra

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1 Introduction

Homological algebra established itself as a separate branch of mathematics around the time of WWII. Nowadays it is a profound branch of mathematics and an essential tool. For example, the study of class field theory relies crucially on homological algebra.

An example is the following. Let G is a group. We want to study representations $\rho : G \to \mathrm{Aut}(V)$ where V is a k-module, k some commutative ring. Equivalently, we want to study left $k[G]$-modules. There is a functor $V \mapsto V^G$ which sends V to the invariant submodule $\{v : gv = v \forall g \in G\}$. This functor is representable; in fact, it is just $\mathrm{Hom}(1, -)$ where 1 is the trivial module.

(Some asides. The category of left $k[G]$-modules is enriched over k-modules, so in particular it is pre-additive. It also admits direct sums (that is, biproducts), so it is additive.)

The invariant functor $V \mapsto V^G$ is a functor $k[G]$-Mod $\to$ k-Mod, but since there is a natural inclusion k-Mod $\to$ $k[G]$-Mod we may regard it as a functor $k[G]$-Mod $\to$ $k[G]$-Mod. This functor is additive, and in fact it preserves limits, but it does not preserve short exact sequences. There is a dual functor, the functor $V \mapsto V_G$ of coinvariants, given by $V/\mathrm{span}(gv - v \forall g \in G)$. This functor can also be written $- \otimes 1$. There is an adjunction

$$\mathrm{Hom}(V \otimes 1, W) \cong \mathrm{Hom}(V, \mathrm{Hom}(1, W)) \tag{1}$$

showing that $- \otimes 1$ preserves colimits, but it also does not preserve short exact sequences. Homological algebra in some sense repairs this failure of exactness by associating to a functor a sequence of derived functors. In this particular case we obtain group homology and group cohomology. In general derived functors give us many other examples of homology and cohomology.

In this course we will aim towards some modern developments, e.g. derived categories, exact categories, triangulated categories.

2 Reflections

Let $F : C \to D$ be a functor and let $d \in D$ be an object (both fixed). Consider pairs (d', η) where $d' \in \mathrm{Ob}(C)$ and η is a morphism $\eta : d \to F(d')$ in D. This data induces a map

$$\tau_c : \mathrm{Hom}_C(d', c) \ni \alpha \mapsto F(\alpha) \circ \eta \in \mathrm{Hom}_D(d, F(c)) \tag{2}$$

for every $c \in C$. This is natural in c so we get a natural transformation of contravariant functors $\tau : \mathrm{Hom}_C(d', -) \to \mathrm{Hom}_D(d, F(-))$.

Definition A pair (d', η) is a *reflection* of d in C (along F) if all of the above maps are bijections for all $c \in C$. Equivalently, the natural transformation is a natural isomorphism.

By the Yoneda lemma, if a reflection exists it is unique up to unique isomorphism. (A reflection is precisely a representing object of $\mathrm{Hom}_C(d, F(-))$ in C together with the induced morphism $\mathrm{Hom}_C(d', d') \ni \mathrm{id}_{d'} \mapsto \eta \in \mathrm{Hom}_D(d, F(d'))$.)

Let $\phi : d \to e$ be a morphism and let (d', η_d) and (e', η_e) be reflections of d, e respectively. By uniqueness, there is a unique morphism $\phi' : d' \to e'$ such that the diagram

$$\begin{tikzcd} d \arrow[r, "\eta_d"] \arrow[d, "\phi"'] & F(d') \arrow[d, "F(\phi')"'] \\ e \arrow[r, "\eta_e"'] & F(e') \end{tikzcd} \tag{3}$$

commutes. This induces a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_C(d', c) \arrow[r, "\tau_c"] & \mathrm{Hom}_D(d, F(c)) \\ \mathrm{Hom}_C(e', c) \arrow[r, "\tau_c"] \arrow[u, "(\phi')^*"'] & \mathrm{Hom}_D(e, F(c)) \arrow[u, "\phi^*"'] \end{tikzcd} \tag{4}$$

for every c (in fact of functors).

Suppose we are given that every $d \in D$ has a reflection d'. Then we can write down a functor $G : D \to C$ given by $G(d) = d'$ and $G(\phi) = \phi'$, and the morphisms η_d give a natural transformation $\eta : \mathrm{id}_D \to F \circ G$. The statement that the maps τ are all bijective is now the statement that this is actually the unit of an adjunction

$$\mathrm{Hom}_C(G(d), c) \cong \mathrm{Hom}_D(d, F(c)) \tag{5}$$

where the bijection (natural in c and d) is determined by acting by F, giving a map $\mathrm{Hom}_D(F \circ G(d), F(c))$, then precomposing with η.

The counit $G \circ F \to \mathrm{id}_C$ gives a family of maps $F(G(c)) \to c$ (*coreflections* of objects in C along G).

3 Adjoint functors

Conversely, let $F : C \to D$ and $G : D \to C$ be a pair of adjoint functors (F right adjoint to G) and let

$$\theta_{c,d} : \mathrm{Hom}_C(G(d), c) \to \mathrm{Hom}_D(d, F(c)) \tag{6}$$

be the corresponding family of bijections. Naturality in c, d is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(G(d'), c) \arrow[r, "\theta_{c,d'}"] \arrow[d, "G(\psi)^*"'] & \mathrm{Hom}_D(d', F(c)) \arrow[d, "\psi^*"'] \\
\mathrm{Hom}_C(G(d), c) \arrow[r, "\theta_{c,d}"] \arrow[d, "\phi_*"'] & \mathrm{Hom}_D(d, F(c)) \arrow[d, "F(\psi)_*"'] \\
\mathrm{Hom}_C(G(d), c') \arrow[r, "\theta_{c',d}"] & \mathrm{Hom}_D(d, F(c'))
\end{tikzcd} \tag{7}$$

where $\psi : d' \to d$ and $\phi : c \to c'$ are arbitrary morphisms. In particular, we can define the unit

$$\eta_d = \theta_{G(d),d}(\mathrm{id}_{G(d)}) \in \mathrm{Hom}_D(d, F \circ G(d)) \tag{8}$$

and the counit

$$\varepsilon_c = \theta^{-1}_{c,F(c)}(\mathrm{id}_{F(c)}) \in \mathrm{Hom}_C(G \circ F(c), c) \tag{9}$$

of the adjunction. If $\alpha : G(d) \to c$ is an arbitrary map, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_C(G(d), G(d)) & \xrightarrow{\theta_{G(d),d}} & \mathrm{Hom}_D(d, F \circ G(d)) \\
{\scriptstyle \alpha_*} \downarrow & & \downarrow {\scriptstyle F(\alpha)_*} \\
\mathrm{Hom}_C(G(d), c) & \xrightarrow{\theta_{c,d}} & \mathrm{Hom}_D(d, F(c))
\end{array}
\tag{10}
$$

and examining the image of $\mathrm{id}_{G(d)}$ on the top left in the bottom right, we conclude that

$$\theta_{c,d}(\alpha) = F(\alpha)_* \circ \eta_d. \tag{11}$$

It follows that morphisms $\beta : d \to F(c)$ can be written uniquely as $F(\theta_{c,d}^{-1}(\beta)) \circ \eta_d$, so indeed the adjunction provides reflections. In other words, every morphism $\beta : d \to F(c)$ factors uniquely through $\eta_d : d \to F \circ G(d)$.

Exercise 3.1. *Show that if G is left adjoint to F, then the composite*

$$G(d) \xrightarrow{G(\eta_d)} GFG(d) \xrightarrow{\varepsilon_{G(d)}} G(d) \tag{12}$$

is the identity. Dually,

$$F(c) \xrightarrow{\eta_{F(c)}} FGF(c) \xrightarrow{F(\varepsilon_c)} F(c) \tag{13}$$

is the identity. As an identity of natural transformations, $\varepsilon(G) \circ G(\eta) = id_G$ and $F(\varepsilon) \circ \eta(F) = id_F$.

4 Special cases

Let C be a category and I be a small category. Let C^I denote the category of functors $I \to C$; we call this a diagram of shape I in C. There is a diagonal functor

$$\Delta : C \to C^I \tag{14}$$

sending every $c \in C$ to the constant diagram $I \to C$ with value c (all morphisms are sent to id_c). When I is non-empty, Δ identifies C with a subcategory of C^I. When I is connected, Δ identifies C with a full subcategory of C^I.

(Aside: it is straightforward to produce subcategories which are not full. In fact there is a functor Cat $\to$ Gpd sending a small category to the groupoid of its invertible morphisms, which is a non-full subcategory in general.)

To see this, recall that a morphism $\Delta_c \to \Delta_{c'}$ is a natural transformation, so a family of morphisms $\varphi_i : \Delta_c(i) \to \Delta_{c'}(i)$ such that the squares

$$\begin{tikzcd} \Delta_c(i) \arrow[r, "\varphi_i"] \arrow[d, "\Delta_c(f)"'] & \Delta_{c'}(i) \arrow[d, "\Delta_{c'}(f)"'] \\ \Delta_{c'}(i') \arrow[r, "\varphi_{i'}"] & \Delta_{c'}(i') \end{tikzcd} \tag{15}$$

commute, where $f : i \to i'$ is a morphism in I. But $\Delta_c(i) = c, \Delta_{c'}(i) = c', \Delta_c(f) = \mathrm{id}_c, \Delta_{c'}(i) = \mathrm{id}_{c'}$, so it follows that in fact$\varphi_i = \varphi_{i'}$ whenever a morphism exists from i to i'. To say that I is connected is to say that any pair of objects is connected by a chain of morphisms, so φ_i is constant in i.

When I is empty, C^I is the terminal category 1, and the diagonal functor is the unique functor $C \to 1$.

In general, let D be a diagram (an object in C^I). A reflection of D is a *colimit* (direct limit, generalizing inductive limit) of the diagram, denoted $\varinjlim$. A coreflection of D is a *limit* (inverse limit, generalizing projective limit) of the diagram, denoted $\varprojlim$. Both are unique up to unique isomorphism. Functorial reflection means that Δ has a left adjoint, the colimit functor, and functorial coreflection means that Δ has a right adjoint, the limit functor.

(An equivalent description is that limits and colimits are terminal and initial objects in the categories of cones and cocones over the diagram, where a cone is a morphism $\Delta_c \to D$ and a cocone is a morphism $D \to \Delta_c$.)

Example Let $I = 2$ be the discrete category with two objects. A diagram $2 \to C$ is a pair of objects c_0, c_1. A colimit is then a coproduct $c_0 \sqcup c_1$ and a limit is then a product $c_0 \times c_1$. If I is replaced by a more general discrete category we get more general coproducts and products.

Example Let $I = \{\bullet \rightrightarrows \bullet\}$ be the pair of parallel arrows. A diagram $I \to C$ is a pair of parallel morphisms $f, g : c_0 \to c_1$. A colimit is then a coequalizer and a limit is then an equalizer. In an Ab-enriched category these are equivalently given by the cokernel and kernel of $f - g$.

Example Let I be the category

$$
\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd} . \tag{16}
$$

A limit of a diagram of this shape is a pullback, and the resulting square

$$
\begin{tikzcd}
\varprojlim \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd} \tag{17}
$$

is called a Cartesian square. ($\varprojlim$ is sometimes called the fiber product $X \times_Z Y$ in this context, especially if we work in the category of sets or a similar category.) Dually we can talk about pushouts and co-Cartesian squares.

Classical homological algebra may be regarded as the study of the failure of certain functors to preserve finite limits or colimits.

5 Chain complexes

Read chapter 1 of Weibel! (There are mistakes.)

Let A be an additive category (enriched over Ab with finite biproducts).

Definition A *chain complex* $(C_\bullet, \partial_\bullet)$ in A is a sequence $C_q, q \in \mathbb{Z}$ of objects in A together with a sequence $\partial_q : C_q \to C_{q-1}$ of morphisms (the *boundary* morphisms) such that $\partial_q \circ \partial_{q-1} = 0$ for all q. A *cochain complex* $(C^\bullet, d^\bullet)$ in A is a sequence $C^q, q \in \mathbb{Z}$ of objects in A together with a sequence $d^q : C^q \to C^{q+1}$ of morphisms (the *coboundary* morphisms or the *differentials*) such that $d^{q+1} \circ d_q = 0$ for all q.

When A is a concrete category (e.g. a category of modules), elements of C_q resp. C^q are called chains resp. cochains. A chain complex is precisely an additive functor into A from the preadditive (Ab-enriched) category Γ

$$
... -1 \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow ... \tag{18}
$$

whose objects are the integers $\mathbb{Z}$ such that $\mathrm{Hom}(p, q) = \mathbb{Z}$ if $p - q = 0, 1$ and 0 otherwise, and such that all nontrivial composites are zero. A cochain complex is an additive functor into A from the opposite of this category (e.g. a contravariant additive functor into A from this category). This defines the category of chain complexes as the (additive) functor category $\Gamma \Rightarrow A$; morphisms of chain complexes are natural transformations, or more explicitly maps $f_q : C_q \to D_q$ which are compatible with boundary maps in the sense that $\partial_{D,q} f_q = f_{q-1} \partial_{C,q}$.

There is a collection of functors $[i], i \in \mathbb{Z}$ on Γ (the shifts) which acts on objects by sending q to $q + i$. These are automorphisms. They induce functors from the category of chain complexes to itself as follows: $(C_\bullet, \partial_\bullet)$ is sent to the chain complex $(C[i]_\bullet, \partial[i]_\bullet)$ where $C[i]_q = C_{q-i}, \partial[i]_q = (-1)^i \partial_{q-i}$. The corresponding shifts on cochain complexes takes the form $C[i]^q = C^{q+i}$ and $d[i]^q = (-1)^i d^{q+i}$. The extra signs are a manifestation of the Koszul sign rule, since the shift and the boundary / coboundary maps both have degree 1 and we switch them with each other i times.

(The contravariant functor $q \mapsto -q$ on Γ is a contravariant equivalence. Consequently, the categories of chain complexes and cochain complexes in A are isomorphic, and the isomorphism intertwines the shifts above.)

6 Homology

Let C be a category.

Definition A morphism $a \xrightarrow{\mu} b$ is a *monomorphism*, or *monic*, if for any parallel pair $\phi, \psi : x \rightrightarrows a$, we have $\mu \circ \phi = \mu \circ \psi \Rightarrow \phi = \psi$. Dually, a morphism $a \xrightarrow{\mu} b$ is an *epimorphism*, or *epic*, if for any parallel pair $\phi, \psi : b \rightrightarrows x$, we have $\phi \circ \mu = \psi \circ \mu \Rightarrow \phi = \psi$.

In the category of sets, the monomorphisms are precisely the injections and the epimorphisms are precisely the surjections.

Definition Consider any parallel pair $\alpha, \beta : b \rightrightarrows c$ as a diagram of shape $\{\bullet \rightrightarrows \bullet\}$. A limit of this diagram is an *equalizer*.

Exercise 6.1. *Equalizers are monomorphisms.*

Exercise 6.2. *In the category CRing of commutative unital rings, an epimorphism $\varphi : A \to B$ is a homomorphism of rings such that the multiplication map $B \times B \to B$ induces an isomorphism $B \otimes_A B \to B$.*

Exercise 6.3. *In the category of Hausdorff topological spaces, a morphism $f : X \to Y$ is an epimorphism if $f(X)$ is dense in Y.*

Assume that C has zero objects. Then for $a, b \in C$ there is a distinguished morphism $0 : a \to 0 \to b$, the zero morphism.

Definition The *kernel* $\ker(\alpha)$ of a morphism $a \xrightarrow{\alpha} b$ is the equalizer of $\alpha, 0$. The *cokernel* $\operatorname{coker}(\alpha)$ of a morphism is the coequalizer of $\alpha, 0$.

In particular, kernels are equalizers and cokernels are coequalizers. In an Ab-enriched category, the converse is true: every equalizer is a kernel and every coequalizer is a cokernel (subtract the morphisms).

Definition An additive category A is *abelian* if equalizers and coequalizers exist, every monomorphism is an equalizer, and every epimorphism is a coequalizer.

This is equivalent to requiring that kernels and cokernels exist, that every monomorphism is a kernel, and that every epimorphism is a cokernel. This gives us kernel and cokernel functors.

In a category with zero objects where every morphism has a kernel and a cokernel, given a morphism α we can canonically construct a diagram

$$\begin{tikzcd} \bullet \arrow[r, "\ker(\alpha)"] & \bullet \arrow[r, "\alpha"] \arrow[d, "\operatorname{coker}(\ker(\alpha))"'] & \bullet \arrow[r, "\operatorname{coker}(\alpha)"] & \bullet \\ & \bullet & \bullet \arrow[u, "\ker(\operatorname{coker}(\alpha))"'] & \end{tikzcd} . \tag{19}$$

Write $\operatorname{coker}(\ker(\alpha)) = \operatorname{coim}(\alpha)$ (the coimage) and $\ker(\operatorname{coker}(\alpha)) = \operatorname{im}(\alpha)$ (the image). Using the universal properties of kernels and cokernels we obtain two maps on the bottom making the above diagram commute, and using the fact that certain maps in the above diagram are monomorphisms and epimorphisms, these two maps are equal. So any morphism admits a factorization

$$\alpha = \operatorname{im}(\alpha) \circ \alpha' \circ \operatorname{coim}(\alpha). \tag{20}$$

Exercise 6.4. *Find examples where α' is not an isomorphism.*

Now assume that every monomorphism is a kernel and every epimorphism is a cokernel. Then (probably?) α' is an isomorphism.

Exercise 6.5. *In any category, the following are equivalent:*

1. *α is an isomorphism.*

2. *α is an equalizer and an epimorphism.*

3. *α is a coequalizer and a monomorphism.*

Certainly α' is an isomorphism if in addition C is an additive category. (This gives one definition of an abelian category: an additive category with kernels and cokernels in which α' is always an isomorphism.)

Now let $(C_\bullet, \partial_\bullet)$ be a chain complex. Consider the diagram

$$\begin{tikzcd}
C_{q-1} & C_q \arrow[l, "\partial_q"'] & C_{q+1} \arrow[l, "\partial_{q+1}"'] \arrow[d, "\tilde{\partial_{q+1}}"] \\
& Z_q \arrow[u, "k_q"] & B_q \arrow[ul, "\iota_q"]
\end{tikzcd} \tag{21}$$

where $k_q : Z_q \to C_q$ is the kernel of ∂_q and $\iota_q : B_q \to C_q$ is the image of ∂_{q+1}. We call Z_q the *q-cycles* and B_q the *q-boundaries.*

Exercise 6.6. *If $h = f \circ g$ is a monomorphism, then g is a monomorphism. If h is an epimorphism, then f is an epimorphism.*

Because $\partial_q \circ \partial_{q+1} = 0$ and because B_q is a kernel, we get additional arrows

$$\begin{tikzcd}
& \mathrm{coker}(\partial_{q+1}) \arrow[dl] & \\
C_{q-1} & C_q \arrow[l, "\partial_q"'] \arrow[u] & C_{q+1} \arrow[l, "\partial_{q+1}"'] \arrow[d, "\tilde{\partial_{q+1}}"] \\
H_q & Z_q \arrow[l, "\mathrm{coker}(\tilde{\iota_q})"] \arrow[u, "k_q"] & B_q \arrow[l, "\tilde{\iota_q}"] \arrow[ul, "\iota_q"]
\end{tikzcd} \tag{22}$$

where the *homology* H_q is given by the cokernel of $\tilde{\iota_q}$. So we obtain a sequence of functors $H_q : \mathrm{Ch}(A) \to A$ assuming only that A has zero objects, kernels, and cokernels.

Definition A chain complex $(C_\bullet, \partial_\bullet)$ is *acyclic* if $H_q(C_\bullet) = 0$ for all q.

Definition A chain complex $(C_\bullet, \partial_\bullet)$ is *contractible* if there exists a sequence of arrows $h_q : C_q \to C_{q+1}$ such that the *supercommutator*

$$[\partial, h] = \partial \circ h + h \circ \partial \tag{23}$$

is the identity $\mathrm{id}_{C_\bullet}$. (The supercommutator is the correct commutator to take because ∂ and h have degree -1 and $+1$ respectively.)

Exercise 6.7. *If C is contractible then $H_q(C) = 0$ for all q.*

The converse is false in general.

Exercise 6.8. *If C is a chain complex, show that cone($\mathrm{id}_{C_\bullet}$) is contractible, where the* cone *is defined as follows: if $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, cone(f) is a chain complex with*

$$cone(f)_q = C_q \oplus D_{q+1} \tag{24}$$

and

$$\partial_q^{cone(f)} = \begin{bmatrix} \partial_q^C & 0 \\ f_q & -\partial_{q+1}^D \end{bmatrix}. \tag{25}$$

It is a nice exercise to show that if A is an abelian category then so is $\mathrm{Ch}(A)$. Furthermore, a chain map is a monomorphism if and only if it is pointwise a monomorphism, and similarly for epimorphisms. Warning: this is false for the category of short exact sequences in A. An explicit example is the morphism

$$\begin{tikzcd} 0 & \mathbb{Z}/4\mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d, "\mathrm{id}_\mathbb{Z}"] & \mathbb{Z} \arrow[l, "4"'] \arrow[d, "2"] & 0 \arrow[l] \\ 0 & \mathbb{Z}/2\mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "2"'] & 0 \arrow[l] \end{tikzcd} \tag{26}$$

which has trivial kernel but is not pointwise a monomorphism.

Let $F : A \to B$ be an additive functor between abelian categories. Then we get a diagram of functors

$$\begin{tikzcd} \mathrm{Ch}(A) \arrow[r, "F"] \arrow[d, "H_q"] & \mathrm{Ch}(B) \arrow[d, "H_q"'] \\ A \arrow[r, "F"] & B \end{tikzcd} \tag{27}$$

and this diagram does not commute in general; we will be studying this. In the special case that we want to study the action of F on an acyclic complex, this is equivalent to studying the action of F on short exact sequences (an acyclic complex breaks up into short exact sequences), and this leads to the following definition.

Definition An additive functor $F : A \to B$ is *left exact* if for any short exact sequence $0 \to a \to b \to c \to 0$, the sequence $0 \to F(a) \to F(b) \to F(c)$ remains exact. Dually, it is *right exact* if $F(a) \to F(b) \to F(c) \to 0$ remains exact. (For chain complexes we should turn these arrows around.)

To any additive functor $F : A \to B$ we will associate a sequence of functors $R^q(F)$ and $L_q(F)$, the right and left derived functors. There are canonical natural transformations $F \to R^0(F)$ and $L_0(F) \to F$, and F is left exact iff $R^q(F)$ is an isomorphism iff $R^q(F) = 0$ for all $q \geq 1$. Dually, F is right exact iff $L_0(F) \to F$ is an isomorphism iff $L_q(F) = 0$ for all $q \geq 1$.

An important point is that although F does not preserve acyclicity, it preserves contractibility (which is one way to see that the two are not equivalent). A special kind of complex (projective objects bounded below or injective objects bounded above) has the property that if it is acyclic, it is automatically contractible, and these will be important: we will try to replace objects of abelian categories with complexes homotopy equivalent to them.

Definition A chain map $f_\bullet : C_\bullet \to D_\bullet$ is *null-homotopic* if there exist $h_q : C_q \to D_{q+1}$ such that

$$[\partial, h] = \partial^C_{q+1} \circ h_q + h_{q-1} \circ \partial^C_q = f. \tag{28}$$

Two chain maps f, g are *chain homotopic* if $f - g$ is null-homotopic, and a chain map f is a *chain homotopy equivalence* if there is a map g in the other direction such that fg, gf are both chain homotopic to identities.

Exercise 6.9. *Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map. Show that in the diagram*

$$\begin{tikzcd} & C_\bullet \arrow[dl, "f"'] & \\ D_\bullet & cone(id_D) \arrow[l] & D[-1]_\bullet \arrow[l] \end{tikzcd} \tag{29}$$

it is possible to lift f to a chain map $C_\bullet \to cone(id_D)$ if and only if f is null-homotopic. The set of liftings is in one-to-one correspondence with the set of contracting homotopies of f.

Exercise 6.10. *If an epimorphism is the cokernel of some morphism, and its kernel exists, then it is the cokernel of its kernel.*

7 Homological algebra

Let A be an abelian category. We will prove three fundamental lemmas that encapsulate much of homological algebra.

Definition An object P of A is *projective* if for any diagram of the form

$$\begin{tikzcd} P \arrow[d, "f"'] \arrow[dr, dotted, "\tilde{f}"] & \\ Q' & Q \arrow[l, two heads, "p"] \end{tikzcd} \tag{30}$$

with p an epimorphism, there exists a morphism (not necessarily unique) $\tilde{f} : P \to Q$ making the above diagram commute. We say that $\tilde{f}$ is a *lift* of f to Q.

Definition Dually, an object I of A is *injective* if for any diagram of the form

$$\begin{tikzcd} I & \\ Q' \arrow[u, "f"] \arrow[r, tail, "p"'] & Q \arrow[ul, dotted, "\tilde{f}"'] \end{tikzcd} \tag{31}$$

with p a monomorphism, there exists a morphism (not necessarily unique) $\tilde{f} : Q \to I$ making the above diagram commute. We say that $\tilde{f}$ is an *extension* of f to Q.

Extension also has another meaning. Namely, we say that a short exact sequence of the form

$$Q'' \twoheadleftarrow Q \leftarrowtail Q' \tag{32}$$

exhibits Q as an extension of Q'' by Q'. We may construct a category $\mathrm{Ext}(Q'', Q')$ whose objects are all such extensions (Q'', Q' fixed, Q varying) and whose morphisms are commutative diagrams (the maps between copies of Q'', Q' being identities). When we have finite biproducts, we can define the Baer sum of two extensions. This sum operation descends to isomorphism classes and can be used to define the Ext group (at least in an abelian category), which can also be obtained by considering the derived functors of Hom functors.

Definition An abelian category *has enough projectives* if for every object M there exists an epimorphism $P \twoheadrightarrow M$ where P is projective. Dually, it *has enough injectives* if for every object M there exists a monomorphism $M \rightarrowtail I$ where I is injective.

We now assume that A has enough projectives. This is always true, for example, in categories of modules, where free modules are projective. More generally, if $F : C \to \mathrm{Set}$ is a functor with a left adjoint $G : \mathrm{Set} \to C$, then $G(X)$ is projective; morphisms $G(X) \to Q$ are identified with functions $X \to F(Q)$, and such functions can clearly be lifted. Any object M therefore admits an epimorphism $G(F(M)) \to M$.

Definition A *projective resolution* of an object M is an acyclic chain complex

$$\cdots 0 \leftarrow M \xleftarrow{\partial_0} P_0 \xleftarrow{\partial_1} P_1 \cdots \tag{33}$$

with all P_i projective.

This is a common definition, but actually a projective resolution should be thought of as the following, and the above is an *augmented* resolution.

Definition A *projective resolution* of an object M is a quasi-isomorphism (induces isomorphisms on homology)

$$\begin{tikzcd}
0 \arrow[d] & P_0 \arrow[l] \arrow[d] & P_1 \arrow[l, "\partial_1"'] \arrow[d] & P_2 \arrow[l, "\partial_2"'] \arrow[d] & \cdots \arrow[l] \\
0 & M \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{34}$$

from a non-negatively graded chain complex all of whose components are projective to M.

Dually we can define injective resolutions.

Theorem 7.1. *(First fundamental lemma) For any diagram in an abelian category of the form*

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\partial_0"'] & P_1 \arrow[l, "\partial_1"'] & P_2 \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\ 0 & N \arrow[l] & Q_0 \arrow[l, "\partial'_0"] & Q_1 \arrow[l, "\partial'_1"] & Q_2 \arrow[l, "\partial'_2"] & \cdots \arrow[l] \end{tikzcd} \tag{35}$$

where the bottom and top row are chain complexes, the P_i are projective, and the bottom row is acyclic, there exist morphisms $f_i : P_i \to Q_i$ making the above diagram commute.

Proof. We proceed inductively. The composition $f \circ \partial_0$ is a map to the image of an epimorphism (by acyclicity). By the projectivity of P_0, there exists a lift $f_0 : P_0 \to Q_0$ making the diagram commute:

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\partial_0"'] \arrow[d, dotted, "f_0"] & P_1 \arrow[l, "\partial_1"'] & P_2 \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\ 0 & N \arrow[l] & Q_0 \arrow[l, "\partial'_0"] & Q_1 \arrow[l, "\partial'_1"] & Q_2 \arrow[l, "\partial'_2"] & \cdots \arrow[l] \end{tikzcd} \tag{36}$$

The composite $\partial'_0 \circ f_0 \circ \partial_1 : P_1 \to N$ is 0 by commutativity, so $f_0 \circ \partial_1$ factors through the kernel of ∂'_0, hence by acyclicity through the image of ∂'_1. But the map from Q_1 to $\mathrm{im}(\partial'_1)$ is an epimorphism, so by the projectivity of P_1 there exists a lift $f_1 : P_1 \to Q_1$ making the diagram commute:

$$\begin{tikzcd} 0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l, "\partial_0"'] \arrow[d, dotted, "f_0"] & P_1 \arrow[l, "\partial_1"'] \arrow[d, dotted, "f_1"] & P_2 \arrow[l, "\partial_2"'] & \cdots \arrow[l] \\ 0 & N \arrow[l] & Q_0 \arrow[l, "\partial'_0"] & Q_1 \arrow[l, "\partial'_1"] & Q_2 \arrow[l, "\partial'_2"] & \cdots \arrow[l] \end{tikzcd} \tag{37}$$

The rest of the argument is the same. $\square$

Dually we get a corresponding result for complexes with injective terms:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "d_0"] & I_0 \arrow[r, "d_1"] & I_1 \arrow[r, "d_2"] & I_2 \arrow[r] & \cdots \\ 0 \arrow[r] & N \arrow[r, "d'_0"'] \arrow[u, "f"'] & Q_0 \arrow[r, "d'_1"'] \arrow[u, dotted, "f_0"'] & Q_1 \arrow[r, "d'_2"'] \arrow[u, dotted, "f_1"'] & Q_2 \arrow[r] \arrow[u, dotted, "f_2"'] & \cdots \end{tikzcd} \tag{38}$$

Note that if we wanted both of these results for categories of modules we would be forced to prove the projective and injective results separately because the opposite of a category of modules is usually not a category of modules. But we proved the lemma for abelian categories, and the opposite of an abelian category is an abelian category.

Above we lifted non-uniquely infinitely many times. Do these choices matter?

Theorem 7.2. *(Second fundamental lemma) Any two choices f_i, f_i' of liftings making the above diagram commute are chain homotopic as morphisms of chain complexes (including f).*

These two lemmas imply that, although projective resolutions are not unique, a morphism between any two objects M, N extends to a morphism between projective resolutions which is unique up to chain homotopy equivalence, and an isomorphism between any two objects M, N lifts to a chain homotopy equivalence between any two projective resolutions; moreover, this chain homotopy equivalence is itself unique up to chain homotopy.

Proof. It suffices to show by subtraction that any lift of the morphism $f = 0$ is null-homotopic:

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "0"] \arrow[dr, dotted, "0"] & P_0 \arrow[l, "\partial_0"'] \arrow[d, "f_0"] \arrow[dr, dotted, "h_0"] & P_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] \arrow[dr, dotted, "h_1"] & P_2 \arrow[l, "\partial_2"'] \arrow[dr, dotted] & \cdots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l, "\partial_0'"] & Q_1 \arrow[l, "\partial_1'"] & Q_2 \arrow[l, "\partial_2'"] & \cdots \arrow[l]
\end{tikzcd} \tag{39}$$

That is, we want to find a sequence of maps $h_i : P_i \to Q_{i+1}$ such that $f_i = \partial_{i+1}' h_i + h_{i-1}\partial_i$. This is clear for $i = -1$. For $i = 0$ we need to find a map $h_0 : P_0 \to Q_1$ such that $f_0 = \partial_1' h_0$. By commutativity, $\partial_0' \circ f_0 = 0 \circ \partial_0 = 0$, hence f_0 factors through $\ker(\partial_0')$, hence by acyclicity through $\mathrm{im}(\partial_1')$. The map from Q_1 to its image is an epimorphism, so by the projectivity of P_0, the corresponding map lifts to a map to Q_1, which gives our desired h_0.

For general i we proceed inductively. If h_{i-1} has already been found, we want to find h_i such that $\partial_{i+1}' h_i = f_i - h_{i-1}\partial_i$. By commutativity and the assumption that $f_{i-1} = \partial_i' h_{i-1} + h_{i-2}\partial_{i-1}'$, we compute that

$$\begin{aligned}
\partial'_i(f_i - h_{i-1}\partial_i) &= f_{i-1}\partial_i - (\partial'_i h_{i-1})\partial_i && (40)\\
&= f_{i-1}\partial_i - (f_{i-1} - h_{i-2}\partial_{i-1})\partial_i && (41)\\
&= 0. && (42)
\end{aligned}$$

By acyclicity and projectivity, it follows as above that $f_i - h_{i-1}\partial_i$ factors through ∂'_{i+1}, and the conclusion follows. □

8 First approximation to derived functors

Definition A non-unital subcategory I of an Ab-enriched category A is an *ideal* if it has the same objects as A and if its morphisms are closed under addition and under composition by morphisms from A. (In practice A will in fact be enriched over k-Mod for some commutative ring k.) The *quotient category* A/I is the cokernel of the inclusion $I \to A$ in Ab-enriched categories.

Example Let A be the category of chain complexes $\mathrm{Ch}(B)$ over an additive category B and let I be the non-unital subcategory of null-homotopic morphisms. The quotient A/I is the *homotopy category of chain complexes* $K(B)$.

Let F be an additive functor between two additive (?) categories C, D. We may define, as an approximation, the *left derived functor* $LF : K(C) \to K(D)$ to be $K(F) \circ P$ where P is any *projective resolution functor* $P : C \to K(C)$. This is a functor sending objects to projective resolutions; such a functor exists and is determined by its action on objects by the first and second fundamental lemmas. $K(F)$ is well-defined because additive functors preserve null-homotopic morphisms. We can now define a sequence of functors

$$L_q(F) = H_q \circ K(F) \circ P : C \to D. \qquad (43)$$

Moreover, we get a natural transformation $L_0(F) \to F$ coming from the fact that $P : C \to K(C)$ is left adjoint (?) to zeroth homology $H_0 : K(C) \to C$. We say that F is *right exact* if this natural transformation is an isomorphism. Dually we can use cochain complexes, injective resolutions, and cohomology functors to define right derived functors $R(F), R^q(F)$.

This is not quite the correct definition of total derived functor. We really want to define the left and right derived functor on a category which is closer to chain complexes.

We say that an object M is *F-acyclic* if $(L_qF)(M) = 0$ for $q > 0$ and $L_0F(Q)$ is canonically isomorphic to Q (this condition is unnecessary if F is right exact). To compute L_qF it suffices to use an F-acyclic resolution.

9 Bar resolutions and the classical theory of derived functors

10 Double complexes

Let A be an Ab-enriched category. A *double complex* $(C_{\bullet\bullet}, \partial^{\leftarrow}, \partial^{\downarrow})$ is a collection C_{pq} of objects in A ($p, q \in \mathbb{Z}$) together with boundary maps

$$C_{p-1,q} \xleftarrow{\partial^{\leftarrow}_{pq}} C_{p,q} \tag{44}$$

and

$$C_{pq} \xrightarrow{\partial^{\downarrow}_{pq}} C_{p,q-1} \tag{45}$$

such that $(\partial^{\leftarrow} + \partial^{\downarrow})^2 = 0$. More explicitly, every row $(C_{\bullet q}, \partial^{\leftarrow}_{\bullet q})$ and every column $(C_{p\bullet}, \partial^{\downarrow}_{p\bullet})$ is a chain complex, and vertical and horizontal boundaries supercommute:

$$[\partial^{\leftarrow}, \partial^{\downarrow}] = \partial^{\leftarrow}\partial^{\downarrow} + \partial^{\downarrow}\partial^{\leftarrow} = 0. \tag{46}$$

We say that an element of C_{pq} has total degree $p + q$. A portion of a double complex looks like

$$\begin{tikzcd} C_{p,q-1} & C_{pq} \arrow[l, "\partial^{\leftarrow}_{pq}"'] \arrow[d, "\partial^{\downarrow}_{pq}"] \\ & C_{p-1,q} \end{tikzcd} \tag{47}$$

There is a natural generalization to triple complexes, etc. and in each case the condition we want on the differentials $\partial', ...\partial^{(n)}$ is that their sum (the *total differential*) squares to zero. Equivalently, every plane is a double complex. So in some sense nothing new happens after double complexes.

The category of double complexes is isomorphic to the category of complexes of complexes; that is, $\mathrm{Ch}(\mathrm{Ch}(A))$. A complex of complexes gives rise to horizontal and vertical differentials which commute rather than anticommute. Flipping some signs addresses this; this amounts to modifying the notion of degree to be total degree.

Exercise 10.1. *What are the projective objects in Ch(A)? In bounded-below complexes? Bounded-above complexes? Bounded complexes? Assume that A has enough projectives.*

The category of double complexes has a natural involution giving by switching horizontal and vertical directions. This would be more annoying to do with complexes of complexes, where various signs would need to be switched.

Associated to every double complex (in a category where suitable limits and colimits exist) is various *total complexes*, all of whose differentials are given by the total differential $\partial^{\leftarrow} + \partial^{\rightarrow}$. The idea is that we want the n^{th} component to be obtained from combining the elements of the diagonal $C_{pq}, p + q = n$. We can do this using either the direct sum or the product, giving two double complexes

$$\mathrm{Tot}_n^{\oplus} = \bigoplus_{p+q=n} C_{pq} \hookrightarrow \prod_{p+q=n} C_{pq} = \mathrm{Tot}_n^{\Pi}. \tag{48}$$

The direct sum consists of elements of the direct product with finite support. We can also talk about elements with support bounded on the left or with support bounded on the right, which gives two more complexes

$$\mathrm{Tot}_n^{+} = \bigoplus_{p+q=n, q<0} C_{pq} \oplus \prod_{p+q=n, q\geq 0} C_{pq} \tag{49}$$

$$\mathrm{Tot}_n^{-} = \bigoplus_{p+q=n, p<0} C_{pq} \oplus \prod_{p+q=n, p\geq 0} C_{pq} \tag{50}$$

(but the choice to split at 0 is arbitrary).

Many double complexes in practice are supported on a half-plane or even only on the first quadrant $p, q \geq 0$, in which case all of the above total complexes are equivalent. The same is true of the third quadrant. In the second and fourth quadrant, there are two types of total complexes.

Every double complex gives rise to two *spectral sequences.* To get one of them, forget about the horizontal arrows. This gives a collection of columns, each of which are chain complexes. We can compute the homology of all of these complexes, and the horizontal arrows descend to maps on homology. The *zeroth page* or E^0 term of the spectral sequence is

$$E^0_{pq} = C_{pq} \tag{51}$$

and it is equipped with differentials $d^0_{pq} = \partial^{\downarrow}_{pq}$. The *first page* or E^1 term of the spectral sequence is the homologies of the columns

$$E^1_{pq} = H^{\downarrow}_{pq} \tag{52}$$

and it is equipped with differentials d^1_{pq} given by the maps induced on homology by $\partial^{\leftarrow}_{pq}$. The *second page* or E^2 term of the spectral sequence is the homologies of the first page

$$E^2_{pq} = H_p(E^1_{\bullet q}, d^1_{\bullet q}). \tag{53}$$

Elements of E^2_{pq} are represented by vertical cycles whose horizontal boundaries are also vertical boundaries; that is, by elements $z_{pq} \in C_{pq}$ such that $\partial^{\downarrow}_{pq} z_{pq} = 0$ and such that $\partial^{\leftarrow}_{pq} z_{pq} \in \partial^{\downarrow}(C_{p-1,q+1})$. Such an element is said to *survive* to the E^2 term.

E^2 itself has a differential as follows, at least when the underlying abelian category is a category of modules. Write $\partial^{\leftarrow}_{pq} z_{pq} = \partial^{\downarrow} w_{p-1,q+1}$. Then $\partial^{\leftarrow} w_{p-1,q+1} \in C_{p-2,q+1}$ survives to E^2; moreover, as an element $[w_{p-1,q+1}] \in E^2_{p-2,q+1}$ it is unique. This is the differential $d^2_{pq}[z_{pq}]$. This generalizes.

Exercise 10.2. *Construct this differential using only universal constructions; in particular, do not use elements.*

Switching the vertical and horizontal indices gives a second spectral sequence.

In general, E^r_{pq} is a subquotient of C_{pq}. When we can talk about elements, it is represented by elements $w_{pq} \in C_{pq}$ for which there exists a sequence

$$w_{p-1,q+1}, w_{p-2,q+2}, ..., w_{p-r+1,q+r-1} \tag{54}$$

in $C_{p-1,q+1}$, etc. such that

$$\partial^{\downarrow} w_{p-i,q+i} + \partial^{\leftarrow} w_{p-i+1,q+i-1} = 0 \tag{55}$$

for all $r-1$ pairs of neighboring ws, together with the condition $\partial^{\downarrow} w_{pq} = 0$. If we have an infinite such sequence, then we can regard them as describing an element of $\mathrm{Tot}^{+}_{p+q}(C_{\bullet\bullet})$, and this element has zero total differential; that is, it is a cycle of the total complex. If we consider elements $w_{pq} \in C_{pq}$ satisfying the above condition up to a suitable equivalence relation, then E^r_{pq} approximates as $r \to \infty$ the homology of the total complex.

d^r_{pq} takes w_{pq} satisfying the above condition to $\partial^{\leftarrow} w_{p-r,q+r}$. We then define

$$E^{r+1}_{pq} = \mathrm{Ker}(d^r_{pq})/\mathrm{Im}(d^r_{p+r,q-r+1}). \tag{56}$$

From every double complex we therefore obtain a pair of spectral sequences. If the double complex is first-quadrant, then both spectral sequences converge to the homology of the total complex.

Spectral sequences can be used to compute derived functors using F-acyclic objects.

Suppose in that in the total complex $\mathrm{Tot}(C_{\bullet\bullet})$ we are given an element $(w_{pq})_{p+q=n}$ with $w_{ij} = 0$ for $i < p_0$ (so the element belongs to Tot^{-}) and $\partial^{\leftarrow} w_{p_0 q_0} = 0$. Assume furthermore that $(C_{\bullet q_0}, \partial^{\leftarrow})$ is acyclic. Then there exists $v_{p_0+1,q_0} \in C_{p_0+1,q_0}$ such that $\partial^{\leftarrow} v_{p_0+1,q_0} = w_{p_0,q_0}$ and such that, writing

$$v = (0, ...0, v_{p_0+1,q_0}, 0, ...), \tag{57}$$

we have

$$w = \partial^{\mathrm{tot}} v + (0, ...0, 0, \tilde{w}_{p_0+1,q_0-1}, w_{p_0+2,q_0-2}, ...) \tag{58}$$

where $w_{p_0+1,q_0+1} = \partial^{\downarrow} v_{p_0+1,q_0} + \tilde{w}_{p_0+1,q_0-1}$. Thus by subtracting a total differential, we can replace w with another element of the total complex whose support has been moved down one place and which is equal to w two or more places down. If $\partial^{\leftarrow} \tilde{w}_{p_0+1,q_0-1} = 0$, then we can continue this process. This is an archetypal diagram chasing argument.

A sample application: if $C_{\bullet\bullet}$ is a double complex in the first quadrant whose rows and columns are all acyclic except for the leftmost column and bottom row, then the total complex admits two maps, one to the bottom homology complex and one to the left homology complex, both of which are quasi-isomorphisms. This shows that their homologies are isomorphic, but the isomorphism is not induced by a map of chain complexes but by a morphism in the derived category.

11 Long exact sequences

Suppose we have a short exact sequence of chain complexes

$$C''_\bullet \twoheadleftarrow C_\bullet \hookleftarrow C'_\bullet \tag{59}$$

of, say, R-modules. We may regard it as a double complex. If $w_q \in C_q$, then $p : C_\bullet \to C''_\bullet$ sends it to some $z''_q \in C''_q$, and then the boundary sends it to $\partial'' z''_q \in C''_{q-1}$. By commutativity, this is also $p\partial w_q$. Let z'_{q-1} be the unique element of C'_{q-1} such that $i(z'_{q-1}) = \partial w_q$ where i is th emap from $C'_\bullet$ to $C_\bullet$. Then

$$0 = \partial^2 w_q = (i \circ \partial')z'_{q-1} \tag{60}$$

and since i is a monomorphism, $\partial' z'_{q-1} = 0$, hence $z'_{q-1} \in Z'_{q-1}$.

Suppose that $w_q, \tilde{w}_q$ are sent to the same z''_q. Then $p(w_q - \tilde{w}_q) = 0$, so by exactness $w_q - \tilde{w}_q = i(v'_q)$ for some $v'_q \in C'_q$. Then

$$\partial(w_q - \tilde{w}_q) = (\partial \circ i)(v'_q) = i\partial' v'_q. \tag{61}$$

Let $z'_{q-1}, \tilde{z}'_{q-1}$ be the corresponding elements of Z'_{q-1}. Then

$$i(z'_{q-1} - \tilde{z}'_{q-1}) = \partial w_q - \partial \tilde{w}_q = i\partial v'_q \tag{62}$$

and since i is a monomorphism, we conclude that $z'_{q-1} - \tilde{z}'_{q-1} \in B'_{q-1}$.

Consequently the assignment $z''_q \to z'_{q-1}$ gives a well-defined map $Z''_q \to H'_{q-1}$. This map is an R-module homomorphism, as one can verify by choosing lifts appropriately. Moreover, the kernel of this homomorphism contains B''_q, and in fact it is a submodule of $B''_q + p(Z_q)$. In fact the kernel of the map $H''_q \to H'_{q-1}$ (the connecting homomorphism) is the image of $H_q \xrightarrow{p} H''_q$.

Exercise 11.1. *Show that in fact we have a long exact sequence*

$$\cdots H'_{q-1} \leftarrow H''_q \leftarrow H_q \leftarrow H'_q \leftarrow H''_{q+1} \cdots \tag{63}$$

12 A categorical interlude

Let C be a category and a an object. The category $a^{\leftarrowtail}$ of monomorphisms into a is the category whose objects are monomorphisms $\mu : x \rightarrowtail a$ and whose morphisms are commuting triangles.

Exercise 12.1. *Show that $a^{\leftarrowtail}$ is a preorder.*

In other words, if a monomorphism factors through another monomorphism, the factorization is unique. Moreover, any two isomorphic objects in $a^{\leftarrowtail}$ are isomorphic via a unique isomorphism. A *subobject* of a is an isomorphism class of objects in $a^{\leftarrowtail}$.

If C is a preorder and C_0 is a collection of objects, then the lower bounds of C_0 are the elements of C with a morphism to every element of C_0, and the upper bounds of C_0 are the elements of C with a morphism from every element of C_0. An infimum is a terminal object of the lower bounds and a supremum is an initial object of the upper bounds.

Exercise 12.2. *Show that in $a^{\leftarrowtail}$, infimum is the categorical product.*

Exercise 12.3. *Show that in $a^{\leftarrowtail}$, infimum is the categorical limit.*

For a pair of elements $s_1, s_2 \in a^{\leftarrowtail}$ we may write their infimum as $s_1 \cap s_2$; it generalizes the intersection. We can write the infimum as a pullback

$$\begin{tikzcd} s_1 \arrow[d, tail] & s_1 \times_a s_2 \arrow[l] \arrow[d] \arrow[dl, tail] \\ a & s_2 \arrow[l, tail] \end{tikzcd} \tag{64}$$

because the pullback of a monomorphism is a monomorphism. Dually, the pushout of an epimorphism is an epimorphism.

A category is *well-powered* if $a^{\leftarrowtail}$ is essentially small. In this case we can talk about infima of arbitrary collections of subobjects.

An epimorphism $\eta : a \to b$ is *strong* if for every commutative diagram of the form

$$\begin{tikzcd} a \arrow[r, two heads, "\eta"] \arrow[d, "\alpha"'] & b \arrow[d, "\beta"] \arrow[dl, dotted, "\kappa"] \\ x \arrow[r, tail, "\mu"] & y \end{tikzcd} \tag{65}$$

there exists an arrow $\kappa : b \to x$ such that $\alpha = \kappa \circ \eta, \beta = \mu \circ \kappa$.

Exercise 12.4. *Show that in a commutative diagram of the above form together with an arrow $\kappa : b \to x$, we have $\alpha = \kappa \circ \eta$ if and only if $\beta = \mu \circ \kappa$.*

Exercise 12.5. *Show that κ is unique if it exists.*

An epimorphism $\eta : a \to b$ is *extremal* if, whenever η factors through a monomorphism $\nu : c \to b$, then ν is an isomorphism.

Exercise 12.6. *Show that a coequalizer is a strong epimorphism.*

Exercise 12.7. *Show that a strong epimorphism is an extremal epimorphism.*

A *strong epi-mono* factorization of a morphism $\alpha : a \to b$ is a factorization of α into the product of a monomorphism and a strong epimorphism. A category *has strong epi-mono factorization* if every morphism has this property.

Exercise 12.8. *Show that a morphism is an isomorphism if and only if it is a monomorphism and a strong epimorphism.*

Exercise 12.9. *The supremum of a family of subobjects exists if a category has strong epi-mono factorization and coproducts exist.*

Let A be an abelian category. Consider a diagram

$$\begin{tikzcd} M_0 \arrow[d, "f_0"'] & M' \arrow[l, "g_0"] \arrow[d, "g_1"] \\ M'' & M_1. \arrow[l, "f_1"] \end{tikzcd} \tag{66}$$

This square commutes if and only if the sequence

$$M'' \xleftarrow{f_0 - f_1} M_0 \oplus M_1 \xleftarrow{g_0 \oplus g_1} M' \tag{67}$$

is a chain complex. It is cartesian if and only if the sequence

$$M'' \xleftarrow{f_0 - f_1} M_0 \oplus M_1 \xleftarrow{g_0 \oplus g_1} M' \leftarrow 0 \tag{68}$$

is exact, and it is cocartesian if and only if the sequence

$$0 \leftarrow M'' \xleftarrow{f_0 - f_1} M_0 \oplus M_1 \xleftarrow{g_0 \oplus g_1} M' \tag{69}$$

is exact. Hence it is both cartesian an cocartesian if and only if the sequence

$$0 \leftarrow M'' \xleftarrow{f_0 - f_1} M_0 \oplus M_1 \xleftarrow{g_0 \oplus g_1} M' \leftarrow 0 \tag{70}$$

is exact.

Corollary 12.10. *The pullback of an epimorphism is an epimorphism.*

(In other words, abelian categories are regular categories. It is true in any category that the pullback of a monomorphism is a monomorphism.)

That is, let $f_1 : M_1 \to M''$ be an epimorphism and let $f_0 : M_0 \to M''$ be a morphism. The pullback of f_1 along f_0 is a cartesian square

$$\begin{tikzcd} M_0 \arrow[d, "f_0"'] & M_0 \times_{M''} M_1 \arrow[l, "\tilde{f}_1"'] \arrow[d, "\tilde{f}_0"] \\ M'' & M_1. \arrow[l, two heads, "f_1"] \end{tikzcd} \tag{71}$$

and the claim is that $\tilde{f}_1$ is still an epimorphism. This is because the above diagram is actually also cocartesian.

To show that a morphism in an Ab-enriched category is an epimorphism, it suffices to show that if $h \circ \tilde{f}_1 = 0$, then $h = 0$, where $h : M_0 \to X$ is some morphism. Together with the zero map $0 : M_1 \to X$, it follows by cocartesianness that $h, 0$ factor through a map $h'' : M'' \to X$. But since f_1 is an epimorphism, $h'' = 0$, hence $h = 0$.

Exercise 12.11. *Show that the pullback of a short exact sequence is a short exact sequence.*

We now construct the connecting homomorphism in homology without elements. Consider the diagram

$$\begin{tikzcd} C''_{q+1} \arrow[d, "\partial''"'] & C_{q+1} \arrow[l, two heads, "\phi"] \arrow[d, "\partial"] & \\ C''_q \arrow[d, "\partial''"'] & C_q \arrow[l, two heads, "p"] \arrow[d, "\partial"] & C'_q \arrow[l, tail] \arrow[d, "\partial'"] \\ C''_{q-1} & C_{q-1} \arrow[l, two heads, "p"] & C'_{q-1}. \arrow[l, tail] \end{tikzcd} \tag{72}$$

There is a monomorphism $j : Z''_q \to C''_q$, and we pull back the middle by the inclusion $Z''_q \to C''_q$ to get another short exact sequence (of preimages of Z''_q).

(To be continued...)

13 Diagrammatics

Let $f : a \to c, g : b \to c$ be a pair of orphisms and let $h : c \rightarrowtail c'$ be a monomorphism.

Exercise 13.1. *The canonical map $a \times_c b \to a \times_{c'} b$ is an isomorphism.*

Now in an abelian category, consider a commutative diagram

$$\begin{tikzcd}
& M \arrow[dl, "0"'] \arrow[d, "f"] & M' \arrow[l, "\tilde{h}"] \arrow[d, "\tilde{f}"] \\
n'' & N \arrow[l, "g"] & N' \arrow[l, "h"]
\end{tikzcd} \tag{73}$$

such that the bottom row is exact (so $\mathrm{im}(h) = \ker(g)$) and such that the square is cartesian.

Exercise 13.2. *$\tilde{h}$ is an epimorphism.*

(Sketch: write down a new pullback square using $\mathrm{im}(h)$. The corresponding pullback is the same by the previous exercise. Use the fact that pullbacks of epimorphisms are epimorphisms in an abelian category.)

Consider a commutative diagram

$$\begin{tikzcd}
C'' \arrow[d, "f''"] & C \arrow[l, "g"'] \arrow[d, tail, "f"] & C' \arrow[l, "h"'] \arrow[d, two heads, "f'"] \\
D'' & D \arrow[l, "i"'] & D' \arrow[l]
\end{tikzcd} \tag{74}$$

with exact rows, where f is a monomorphism and f' is an epimorphism.

Exercise 13.3. *f'' is a monomorphism.*

Some general comments. Let A, B be two filtered objects in an abelian category; that is, there is an increasing collection of subobjects $F_0 \subset F_1 \subset ... \subset A$ and $F'_0 \subset F'_1 \subset ... \subset B$ whose union is A, B respectively. Then a filtered morphism $A \to B$ (one inducing maps $F_i \to F'_i$ induces a maps on associated graded objects

$$\mathrm{gr}(A) = \bigoplus_n F_n/F_{n-1}, \mathrm{gr}(B) = \bigoplus_n F'_n/F'_{n-1}. \tag{75}$$

If A is a filtered chain complex, then so is its associated graded, and this can be used to construct the E^0 page of a spectral sequence. In general, spectral sequences attempt to compute the homology of a filtered complex using the filtration. Taking different filtrations gives different spectral sequences which (hopefully) converge to the same homology.

Consider a commutative diagram

$$\begin{tikzcd} C'' \arrow[d, tail, "f''"'] & C \arrow[l, "p"'] \arrow[d, "f"] & C' \arrow[l, "i"'] \arrow[d, tail, "f'"'] \\ D'' & D \arrow[l, "q"'] & D' \arrow[l, "j"'] \end{tikzcd} \tag{76}$$

where f', f'' are monomorphisms and the rows are exact.

Lemma 13.4. *f is a monomorphism.*

Proof. Consider the inclusion $k : K \to C$ of the kernel of f into C. By assumption $f \circ k = 0$, hence $q \circ f \circ k = f'' \circ p \circ k = 0$. Since f'' is a monomorphism, $p \circ k = 0$, hence by exactness of the top row k factors through a map $\tilde{k} : K \to C'$. Since $0 = f \circ k = j \circ f' \circ \tilde{k}$ and j and f' are both monomorphisms, we conclude that $\tilde{k} = 0$, hence $K = 0$. $\square$

Dually, if f', f'' are epimorphisms, then f is an epimorphism.

Hence with hypotheses as above, if f', f'' are isomorphisms, then f is an isomorphism.

Lemma 13.5. *(Five lemma) Consider a morphism of complexes*

$$\begin{tikzcd} C_{-2} \arrow[d, tail, "f_{-2}"] & C_{-1} \arrow[l, "\partial_{-1}"'] \arrow[d, "f_{-1}"] & C_0 \arrow[l, "\partial_0"'] \arrow[d, "f_0"] & C_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] & C_2 \arrow[l, "\partial_2"'] \arrow[d, two heads, "f_2"] \\ C'_{-2} & C'_{-1} \arrow[l, "\partial'_{-1}"'] & C'_0 \arrow[l, "\partial'_0"'] & C'_1 \arrow[l, "\partial'_1"'] & C'_2 \arrow[l, "\partial'_2"'] \end{tikzcd} . \tag{77}$$

where $f_{\pm 1}$ are isomorphisms, f_{-2} is a monomorphism, and f_2 is an epimorphism. Then f_0 is an isomorphism.

Proof. We can obtain the above diagram by splicing together the following three commutative diagrams:

$$
\begin{tikzcd}
C_{-2} \arrow[d, tail, "f_{-2}"] & C_{-1} \arrow[l, "\partial_{-1}"'] \arrow[d, "f_{-1}"] & Z_{-1} \arrow[l, tail] \arrow[d, "\tilde{f}_{-1}"] & C_0 \arrow[l, "\partial_0"'] \arrow[d, "f_0"] & Z_0 \arrow[l, tail] \arrow[d, "\tilde{f}_0"] & C_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] & C_2 \arrow[l, "\partial_2"'] \arrow[d, two heads, "f_2"] \\
C'_{-2} & C'_{-1} \arrow[l, "\partial'_{-1}"'] & Z'_{-1} \arrow[l, tail] & C'_0 \arrow[l, "\partial'_0"'] & Z'_0 \arrow[l, tail] & C'_1 \arrow[l, "\partial'_1"'] & C'_2 \arrow[l, "\partial'_2"']
\end{tikzcd}
\quad . \tag{78}
$$

(To be continued...)

□

Aside. Every chain complex $(C_\bullet, \partial_\bullet)$ naturally has two filtrations. The stupid filtration is

$$
(F_p^{\text{stupid}} C)_n = \begin{cases} C_n & \text{if } n \geq p \\ 0 & \text{otherwise} \end{cases} \tag{79}
$$

and the good filtration is

$$
(F_p^{\text{good}} C)_n = \begin{cases} C_n & \text{if } n > p \\ Z_n & \text{if } n = p \\ 0 & \text{otherwise} \end{cases} \quad . \tag{80}
$$

Consider a diagram of the form

$$
\begin{tikzcd}
A'' \arrow[d, "i''"'] & A \arrow[l, "f"'] \arrow[d, tail, "i"'] & A' \arrow[l, "g"'] \arrow[d, tail, "i'"] \\
B'' & B \arrow[l, "h"] \arrow[d, "k"'] & B' \arrow[l, "j"] \arrow[d, "k'"] \\
 & C & C' \arrow[l, tail, "\ell"']
\end{tikzcd} \tag{81}
$$

where $f \circ g = 0$ and the middle row and columns are exact (and ℓ is a monomorphism).

Lemma 13.6. *The top row is exact.*

Proof. Consider the factorization of g through maps $\tilde{g} : A' \to K, m : K \to A$ where $K = \operatorname{Ker}(f)$. We want to show that this is the image factorization of g. Consider the composition $i \circ m : K \to B$ and its pullback $n : K \times_B B' \cong L \to B'$. Let $\tilde{j} : L \to K$ denote the other projection map. By commutativity, $\ell \circ k' \circ n = k \circ i \circ m = 0$, and since ℓ is a monomorphism, $k' \circ n = 0$. By exactness of columns, n factors through a morphism $\tilde{n} : L \to A'$. By commutativity and the universal property of L, we conclude that $i \circ m \circ \tilde{j} = i \circ g \circ \tilde{n} = i \circ m \circ \tilde{g} \circ \tilde{n}$. Since i, m are monomorphisms, we conclude that $\tilde{j} = \tilde{g} \circ \tilde{n}$. Since $\tilde{j}$ is an epimorphism, $\tilde{g}$ is an epimorphism, and the conclusion follows. $\square$

Consider now a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[d, tail, "i''"'] & A \arrow[l, "f"'] \arrow[d, "i"'] & \\
B'' \arrow[d, "p''"'] & B \arrow[l, two heads, "h"'] \arrow[d, "p"] & B' \arrow[l, "j"'] \arrow[d, two heads, "p'"] \\
C'' & C \arrow[l, "k"] & C' \arrow[l, "\ell"]
\end{tikzcd} \tag{82}$$

where rows and columns are exact.

Lemma 13.7. *f is an epimorphism.*

Proof. Let D be the pullback of A, B'', B with maps $\tilde{h} : D \to A$ and $m : D \to B$. Let $n : D \to C$ be the composite $p \circ m$. Let E be the pullback of D, C, B' with maps $\tilde{q} : E \to D, \tilde{n} : E \to B'$. Then

$$p \circ m \circ \tilde{q} = n \circ \tilde{q} = q \circ \tilde{n} = \ell \circ p' \circ \tilde{n} = p \circ j \circ \tilde{n} \tag{83}$$

hence

$$p \circ (m \circ \tilde{q} - j \circ \tilde{n}) = 0. \tag{84}$$

Let $r = m \circ \tilde{q} - j \circ \tilde{n}$. Let F be the pullback of A, B, E with maps $\tilde{r} : F \to A, \tilde{i} : F \to E$. Then

$$
\begin{aligned}
i'' \circ f \circ \tilde{r} &= h \circ \imath \circ \tilde{r} && (85)\\
&= h \circ r \circ \tilde{\imath} && (86)\\
&= h \circ (m \circ \tilde{q} - j \circ \tilde{n}) \circ \tilde{\imath} && (87)\\
&= h \circ m \circ \tilde{q} \circ \tilde{\imath} - h \circ j \circ \tilde{n} \circ \tilde{\imath} && (88)\\
&= h \circ m \circ \tilde{q} \circ \tilde{\imath} && (89)\\
&= i'' \circ \tilde{h} \circ \tilde{q} \circ \tilde{\imath}. && (90)
\end{aligned}
$$

Since i'' is a monomorphism,

$$
f \circ \tilde{r} = \tilde{h} \circ \tilde{q} \circ \tilde{\imath}. \qquad (91)
$$

Since a composition of epimorphisms is an epimorphism, we conclude. □

The hypothesis that p' is an epimorphism can be dropped. Doing this will give us the zigzag lemma.

Consider a diagram of the form

$$
\begin{tikzcd}
 & A \arrow[dl, "0"'] \arrow[d, "f"] & A \times_B B' \arrow[l, two heads, dotted, "\tilde{h}"] \arrow[d, dotted] \\
B'' & B \arrow[l, "g"] & B' \arrow[l, "h"]
\end{tikzcd} \qquad (92)
$$

where $A \times_B B'$ is the pullback, $g \circ h = g \circ f = 0$, and the induced map $A \to H(B)$ is zero.

Then the following modified form of lemma 2 holds: $\tilde{h}$ is an epimorphism.

We will need the following going-up and down construction. Consider a diagram of the form

$$
\begin{tikzcd}
A'' \arrow[d, "i''"] & E \arrow[l, two heads, dotted, "\tilde{h}"'] \arrow[d, dotted] & \bar{B} \arrow[l, tail] \arrow[dl, "\ker(h)"'] \\
B'' & B \arrow[l, two heads, "h"'] \arrow[d, "p"] & B' \arrow[l, "j"'] \arrow[u, two heads] \arrow[dl, "q"] \\
& C \arrow[d, two heads] & \\
& \mathrm{Coker}(q) &
\end{tikzcd}
\quad . \tag{93}
$$

Then there is a well-defined map from E to $\mathrm{Coker}(q)$ which lifts to a map from A to $\mathrm{Coker}(q)$.

Consider now a diagram of the form

$$
\begin{tikzcd}
A'' \arrow[d, tail] & A \arrow[l, "f"'] \arrow[d] & \\
B'' \arrow[d] & B \arrow[l] \arrow[d] & B'' \arrow[l] \arrow[dl, "q"] \\
C'' & C \arrow[l] \arrow[d] & \\
& \mathrm{Coker}(q) &
\end{tikzcd}
\tag{94}
$$

with first column semiexact and second column exact.

Then the following modified form of lemma 7 holds: f is an epimorphism. This can be used to prove the zigzag lemma.

Consider a diagram of the form

$$
\begin{tikzcd}
C_0 \arrow[d, "f_0"] & C' \arrow[l, "g_0"'] \arrow[d, "g_1"'] \\
C'' & C_1 \arrow[l, "f_1"']
\end{tikzcd}
\tag{95}
$$

where C_0, C', C_1 are complexes. This induces a diagram

$$
C'' \xleftarrow{f_0 - f_1} C_0 \oplus C_1 \xleftarrow{(g_0, g_1)} C' \tag{96}
$$

which, as we recall, is a chain complex if and only if the diagram commutes and is a short exact sequence if and only if the diagram is cartesian and cocartesian. Any such short exact sequence induces a long exact sequence in homology

$$\cdots \leftarrow H''_q \leftarrow H^0_q \oplus H^1_q \leftarrow H'_q \leftarrow H''_{q+1} \cdots \tag{97}$$

which specializes to the Mayer-Vietoris sequence, etc.

As an application, if $C_{\bullet\bullet}$ is a double complex, then the square

$$\begin{tikzcd} \mathrm{Tot}^+(C) \arrow[d] & \mathrm{Tot}^b(C) \arrow[l] \arrow[d] \\ \mathrm{Tot}(C) & \mathrm{Tot}^-(C) \arrow[l] \end{tikzcd} \tag{98}$$

is both cartesian and cocartesian, hence we get a long exact sequence

$$\cdots \leftarrow H_q(C) \leftarrow H^+_q(C) \oplus H^-_q(C) \leftarrow H^b_q(C) \leftarrow H_{q+1}(C) \cdots \tag{99}$$

where H_q is the homology of the total complex, H^+_q is the homology of Tot^+, etc.

In particular, if the rows of C are acyclic, then Tot^+ is acyclic, and if the columns are acyclic, then Tot^- is acyclic. If rows and columns are both acyclic, then we conclude that $H^b_q(C) \cong H_{q+1}(C)$.

In particular, let C be an acyclic complex. Then we can form a double complex

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots & C_0 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] & C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \cdots & C_{-1} \arrow[l] \arrow[d] & C_0 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \cdots & C_{-2} \arrow[l] \arrow[d] & C_{-1} \arrow[l] \arrow[d] & C_0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & \vdots & \vdots & \vdots & \end{tikzcd} \tag{100}$$

where the p^{th} diagonal consists of copies of C_p and every map is the corresponding boundary in C. This double complex has acyclic rows and columns, so the above applies to it. But $H^b_q \cong Z_q(C)$, hence $H_{q+1} \cong Z_q(C)$.

14 The third fundamental lemma

Consider a diagram of the form

$$\begin{tikzcd}
M' \arrow[d, tail] & Q_0 \arrow[l, two heads, "\partial'_0"'] \arrow[d] & Q_1 \arrow[l, "\partial'_1"'] \arrow[d] & Q_2 \arrow[l, "\partial'_2"'] \arrow[d] & \cdots \arrow[l, "\partial'_3"'] \\
M \arrow[d, two heads] & P_0 \oplus Q_0 \arrow[d] & P_1 \oplus Q_1 \arrow[d] & P_2 \oplus Q_2 \arrow[d] & \cdots \\
M'' & P_0 \arrow[l, "\partial_0"] & P_1 \arrow[l, "\partial_1"] & P_2 \arrow[l, "\partial_2"] & \cdots \arrow[l, "\partial_3"]
\end{tikzcd} \tag{101}$$

where the top row is acyclic, the bottom row is a chain complex with the P_i projective, and the maps in and out of the biproduct are the canoncial maps. Then there exists a filling of the diagram so that the middle row is a chain complex. The filling consists of maps of the form

$$\begin{bmatrix} \partial_i & 0 \\ -\phi_i & \partial'_i \end{bmatrix} : P_i \oplus Q_i \to P_{i-1} \oplus Q_{i-1} \tag{102}$$

with the ϕ_i defined inductively.

Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ in an abelian category, we can choose resolutions of M' and M'' and construct the above diagram. Applying a functor which is exact in the appropriate direction and taking homology gives the corresponding long exact sequence.

15 Various

Suppose that $C_{\bullet\bullet}$ is a double complex and $\Lambda \subset \mathbb{Z} \times \mathbb{Z}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$ equipped with the product partial order which is left-saturated (downward-closed) in the sense that if $(p, q) \in \Lambda$ and $(p', q') \leq (p, q)$ (so $p' \leq p$ and $q' \leq q$) then $(p', q') \in \Lambda$.

From this data we can write down two double complexes. The first is

$$(C^\Lambda)_{pq} = \begin{cases} C_{pq} \text{ if } (p,q) \in \Lambda \\ 0 \text{ otherwise} \end{cases} \tag{103}$$

and the second is

$$(C_\Lambda)_{pq} = \begin{cases} C_{pq} \text{ if } (p,q) \notin \Lambda \\ 0 \text{ otherwise.} \end{cases} \tag{104}$$

All differentials are inherited from C. The first should be regarded as a subcomplex while the second should be regarded as a quotient complex. There is a short exact sequence

$$0 \leftarrow \mathrm{Tot}(C_\Lambda) \leftarrow \mathrm{Tot}(C) \leftarrow \mathrm{Tot}(C^\Lambda) \leftarrow 0 \tag{105}$$

(and similarly for the other total functors) which induces a long exact sequence in homology. Moreover, if C has exact rows, then we get a quasi-isomorphism $\mathrm{Tot}^+(C_\Lambda) \to \mathrm{Tot}^+(C^\Lambda)[1]$.

$\mathrm{Tot}(C)$ is in fact the cone of a certain morphism. Recall that the cone is defined as follows: if $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, $\mathrm{cone}(f)$ is a chain complex with

$$\mathrm{cone}(f)_q = C_q \oplus D_{q+1} \tag{106}$$

and

$$\partial_q^{\mathrm{cone}(f)} = \begin{bmatrix} \partial_q^C & 0 \\ f_q & -\partial_{q+1}^D \end{bmatrix}. \tag{107}$$

This is in fact the total complex of the double complex formed by $f_\bullet, C_\bullet, D_\bullet$. The cone of a morphism is always an extension

$$0 \leftarrow C \leftarrow \mathrm{cone}(f) \leftarrow D[-1] \leftarrow 0 \tag{108}$$

and f induces the connecting homomorphism in the corresponding long exact sequence in homology. This suggests the following question: when is an extension of chain complexes isomorphic to a cone extension as above?

From any chain complex $C_\bullet$ we may write down two short exact sequences

$$0 \leftarrow H_q \leftarrow Z_q \leftarrow B_q \leftarrow 0 \tag{109}$$

and

$$0 \leftarrow B_{q-1} \leftarrow C_q \leftarrow Z_q. \tag{110}$$

Choose a projective resolution $P_H^{\bullet q}$ of H_q and projective resolutions $P_{\bullet q}^B$ of B_q. This gives a direct sum projective resolution $P_Z^{\bullet q}$ of Z_q and $P_C^{\bullet q}$ of C_q. Totalizing gives a map $P_\bullet \to C_\bullet$ which is not a homotopy equivalence but which is a quasi-isomorphism.

Exercise 15.1. *When is an extension of chain complexes $0 \leftarrow C'' \overset{p}{\leftarrow} C \overset{i}{\leftarrow} C' \leftarrow 0$ isomorphic to a cone extension?*

In other words, when does there exist a map $f : C'' \to C'[1]$ such that the corresponding extension

$$0 \leftarrow C'' \xleftarrow{(\mathrm{id},0)} \mathrm{Cone}(f) \xleftarrow{(0,\mathrm{id})} C' \leftarrow 0 \tag{111}$$

is isomorphic to $0 \leftarrow C'' \leftarrow C \leftarrow C' \leftarrow 0$?

Suppose that $(p, r) : C \to \mathrm{Cone}(f)$ induces such an isomorphism. Then commutativity means

$$\begin{bmatrix} p \\ r \end{bmatrix} \circ \partial = \partial^{\mathrm{Cone}(f)} \circ \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} \partial'' & 0 \\ f & \partial' \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix}. \tag{112}$$

This gives

$$\begin{bmatrix} p \circ \partial \\ r \circ \partial \end{bmatrix} = \begin{bmatrix} \partial'' \circ p \\ f \circ p + \partial' \circ r \end{bmatrix}. \tag{113}$$

Hence $f \circ p + [\partial, r] = 0$ (supercommutator). In other words, $f \circ p$ is nullhomotopic and r is a contracting homotopy for $f \circ p$. Commutativity also implies that $r \circ i = \mathrm{id}_{C'}$.

Given r, the corresponding f, if it exists, is unique. To see this, we compute that

$$[\partial, [\partial, r]] = 0 \tag{114}$$

where $[\cdot, \cdot]$ denotes the supercommutator as usual.

Remark: consider a morphism in the category of short exact sequences of chain complexes. By the five lemma, if the leftmost and rightmost maps are isomorphisms, so is the middle map. By the five lemma and the long exact sequence in homology if the leftmost and rightmost maps are quasi-isomorphisms, so is the middle map. In other words, quasi-isomorphisms are closed under extensions.

However, if the leftmost and rightmost maps are chain homotopy equivalences, the middle map is not guaranteed to be a chain homotopy equivalence. For a counterexample, consider the short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & C \arrow[d, tail, "i"'] & C \arrow[l] \arrow[d, "\mathrm{id}"] \\
A \arrow[d, "\mathrm{id}"'] & B \arrow[l, two heads, "p"'] \arrow[d, two heads, "p"'] & C \arrow[l, tail, "i"'] \arrow[d] \\
A & A \arrow[l, "\mathrm{id}"'] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd} \tag{115}$$

The leftmost and rightmost complexes are contractible. The middle complex is contractible if and only if the short exact sequence $0 \leftarrow A \leftarrow B \leftarrow C \leftarrow 0$ splits.

If C, D are two complexes, we can define a double complex $\mathrm{Hom}_{pq}(C, D) = \mathrm{Hom}(C_{-q}, D_p)$. Define

$$L_\partial = L_{\partial^D} : f \mapsto \partial^d \circ f \tag{116}$$

$$R_\partial = R_{\partial^C} : f \mapsto f \circ \partial^C \tag{117}$$

$$\varepsilon : f \mapsto (-1)^{\deg f} f = (-1)^{p+q} f. \tag{118}$$

Then the differentials are $\partial^{\leftarrow} = L_\partial, \partial^{\downarrow} = R_\partial \circ \varepsilon$. The total differential is the supercommutator $f \mapsto [\partial, f]$, and the corresponding total complex is the Hom complex

$$\mathrm{Hom}_n(C, D) = \prod_{p+q=n} \mathrm{Hom}_{pq}(C, D). \tag{119}$$

This complex has cycles given by chain maps of a given degree and boundaries given by null-homotopic chain maps of a given degree. The corresponding homologies are chain homotopy classes of chain maps of a given degree.

Exercise 15.2. *Show that a chain map $f : C \to D$ is a quasi-isomorphism if and only if Cone(f) is acyclic.*

Suppose that

$$h = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \in \mathrm{Hom}_1(\mathrm{Cone}(f), \mathrm{Cone}(f)) \tag{120}$$

is a contracting homotopy for Cone(f), hence $\mathrm{id}_{\mathrm{Cone}(f)} = [\partial^{\mathrm{Cone}(f)}, h]$. Note that h_{11}, h_{22} have degree 1 and h_{12}, h_{21} have degrees $0, 2$. This gives four conditions

$$\begin{bmatrix} \mathrm{id}_C & 0 \\ 0 & \mathrm{id}_{D[-1]} \end{bmatrix} = \begin{bmatrix} [\partial^c, h_{11}] + h_{12} f & \partial^C h_{12} - h_{12}\partial^D \\ h_{21}\partial^C - \partial^D h_{21} + h_{22} f + f h_{11} & f h_{12} - [\partial^D, h_{22}] \end{bmatrix}. \tag{121}$$

The top right condition asserts that h_{12} is a chain map $D \to C$. Denote h_{12} by g, denote $-h_{11}$ by h^C, and denote h_{22} by h^D. The top left condition asserts that h^C is a contracting homotopy for $gf - \mathrm{id}_C$. The bottom right condition asserts that h^D is a contracting homotopy for $fg - \mathrm{id}_D$. In particular, f is a chain homotopy equivalence; moreover, specifying a homotopy inverse and contracting homotopies is equivalent to specifying three conditions describing a contracting homotopy for Cone(f).

The fourth condition may be rewritten

$$-[\partial, h_{21}] = f h^C - h^D f = [f, h]. \tag{122}$$

We compute that

$$\begin{aligned} \partial^{D[-1]} \circ (f h^C - h^D f) - (f h^C - h^D f) \circ \partial^C &= -\partial^D f h^C + \partial^D h^D f - f h^C \partial^C + h^D f \partial^C && (123) \\ &= -f[\partial^C, h^C] + [\partial^D, h^D] f && (124) \\ &= [[\partial, h], f] && (125) \\ &= -f(gf - \mathrm{id}_C) + (fg - \mathrm{id}_D) f && (126) \\ &= 0. && (127) \end{aligned}$$

In other words, the fourth condition asserts that h_{21} contracts the chain map $fh^C - h^D f : C \to D[-1]$.

So the data of a homotopy inverse to f is not quite the same as the data of a contraction of Cone(f): there is the additional data of the contraction h_{21} above. We will call such a thing a strict homotopy equivalence, and we will call a homotopy equivalence such that some h_{21} exists an exact homotopy equivalence.

Suppose (f, g, h^C, h^D) and $(f, g,' h^C, h^D)$ are two exact homotopy equivalences. Then we compute that $f('h^C - h^C)$ is null-homotopic, hence $gf('h^C - h^C)$ is null-homotopic, hence $'h^C - h^C$ is null-homotopic. We also have $[\partial, h^C] = [\partial,' h^C]$, hence $'h^C - h^C$ is a morphism of chain complexes $C \to C[-1]$.

In other words, we can modify h^C by a null-homotopic chain map $C \to C[-1]$. Similarly, we can modify h^D by a null-homotopic chain map $D \to D[-1]$. Consider the pairing

$$\mathrm{Hom}_{K(A)}(C, C[-1]) \times \mathrm{Hom}_{K(A)}(D, D[-1]) \to \mathrm{Hom}_{K(A)}(C, D[-1]) \tag{128}$$

sending a pair (χ, ϑ) to the class of $f(h^C + \chi) - (h^D + \vartheta)f$. This pairing is nondegenerate in the sense that if one of χ, ϑ is fixed then the map induces an isomorphism.

Suppose (f, g, h^C, h^D) is the data of a chain homotopy equivalence. Let $\chi = -g[f, h]$. Then $(f, g, h^C + \chi, h^D)$ is the data of an exact chain homotopy equivalence. To verify this, we compute that

$$\begin{aligned}
\partial^{\mathrm{Hom}}(-h[f,h]) &= -\partial h[f,h] + h[f,h]\partial && (129)\\
&= (-\partial^D h^D - h^D \partial^D)[f,h] && (130)\\
&= (\mathrm{id}_D - fg)[f,h] && (131)\\
&= [f,h] - fg[f,h] && (132)\\
&= f(h^C - g[f,h]) - h^D f && (133)\\
&= f('h^C) - h^D f && (134)
\end{aligned}$$

where $'h^C = h^C + \chi$.

Exercise 15.3. *With hypotheses as above, show that $gh^D -' h^C g = \partial^{Hom}(h^2 g + gh^2 - hgh)$.*

16 More about projective resolutions and left derived functors

Consider an object M in an abelian category A and consider the category of quasi-isomorphisms $Q_\bullet \to M$ (where M is regarded as a complex concentrated in degree 0) where $Q_i = 0$ for $i < 0$. The homotopy category of this category has an initial object, namely any projective resolution. The identity $M \to M$ is the final object. Any additive functor $F : A \to B$ induces a functor from the homotopy category above to the homotopy category of chain complexes in B.

Abelian categories have a notion of cohomological dimension which can be defined as follows. Given an abelian category A, let A_0 denote the subcategory of projective objects, let A_1 denote the subcategory of subobjects of projective objects, and in general let A_n denote the subcategory of subobjects of the objects in A_{n-1}. Elements of A_n may be thought of as syzygies.

Exercise 16.1. *The projective abelian groups are the free groups.*

Exercise 16.2. *Subgroups of free abelian groups are free.*

The *cohomological dimension* or *global dimension* or *homological dimension* $\mathrm{cd}(A)$ is the smallest positive integer n such that $A = A_n$. If this is not true for any n then the cohomological dimension is ∞. An abelian category A has cohomological dimension 0 if and only if every object is projective, if and only if every short exact sequence splits, if and only if A is semisimple.

Recall that an object Q is F-acyclic if $L_qF(Q) = 0$ for $q > 0$ and $L_0F(Q) \cong Q$.

Theorem 16.3. *Let $Q_\bullet$ be an F-acyclic resolution. Then the canonical morphisms $L_qF(Q_\bullet) \to H_qF(Q_\bullet)$ are isomorphisms.*

17 Generators and cogenerators

Theorem 17.1. *(Baer) An abelian group is an injective object in Ab if and only if it is divisible.*

Proof. Suppose I is injective. Let $a \in I$ and consider the diagram

$$
\begin{tikzcd}
I \\
\mathbb{Z} \arrow[u, "\alpha"] \arrow[r, tail, "n"] & \mathbb{Z}
\end{tikzcd}
\tag{135}
$$

where $\alpha(1) = a$. By injectivity we get an extension $\tilde{\alpha} : \mathbb{Z} \to I$ such that $n\tilde{\alpha}(1) = a$, hence I is divisible.

Conversely, suppose I is divisible. Consider a diagram of the form

$$
\begin{tikzcd}
I \\
A \arrow[u, "\alpha"] \arrow[r, tail] & B
\end{tikzcd}
\tag{136}
$$

and consider the poset of partial extensions (A', α') where $A \subset A' \subset B$ and $\alpha' : A' \to I$ extends α. This poset satisfies the hypotheses of Zorn's lemma since it is non-empty and one can take suprema, so it has a maximal element (A_1, α_1).

If A_1 is not all of B, then there exists $b \in B \backslash A_1$. Let $A' = \mathbb{Z}b + A_1$. If $\mathbb{Z}b \cap A_1 = \{0\}$, then A' is a direct sum, and we can extend α_1, which contradicts maximality. Thus $nb \in A_1$ for some $n \in \mathbb{Z}$, and by divisibility we can find some $i \in I$ such that $ni = \alpha_1(nb)$, so we can still extend α_1, which still contradicts maximality. □

More generally, in the category of R-modules (R a principal ideal domain), to show an object is injective it suffices to show that it is injective for inclusions of ideals into R.

Baer's theorem shows in particular that all $\mathbb{Q}$-vector spaces are injective.

Definition A right R-module Q is *flat* if the functor $Q \otimes_R -$ is exact.

Exercise 17.2. *Show that Q is flat if and only if Q is $- \otimes M$-acyclic for every left R-module M.*

Exercise 17.3. *Show that an abelian group is flat if and only if A is torsion-free.*

Exercise 17.4. *Let R be a ring. Show that the following conditions are equivalent:*

1. *Every submodule of a flat left R-module is flat.*

2. *Every submodule of a projective left R-module is flat.*

3. *Every submodule of a free left R-module is flat.*

4. *Every left ideal of R is flat.*

Note that free implies projective implies flat for left R-modules.

Theorem 17.5. *An R-module is flat if and only if it is a filtered colimit of free finite rank modules.*

Exercise 17.6. *Let D be a division algebra which is finite-dimensional over its center F. Show that $D \otimes_F D^{op}$ is a matrix algebra over F.*

A *generator* of a category C is an object g such that $\mathrm{Hom}(g, -)$ is faithful. A *cogenerator* of C is a generator of C^{op}.

Exercise 17.7. *If C has arbitrary coproducts, then g is a generator if and only if for any $c \in C$ there exists an epimorphism $\sqcup_{\gamma \in Hom(g,c)} g \to c$. Dually, if C has arbitrary coproducts, then g is a cogenerator if and only if for any $c \in C$ there exists a monomorphism $c \to \prod_{\gamma \in Hom(c,g)} g$.*

Exercise 17.8. *In an abelian category with a projective generator (resp. injective cogenerator) and arbitrary coproducts (resp. arbitrary products), every object is a quotient of a projective object (resp. subobject of an injective object). In particular, any such category has enough projectives (resp. enough injectives).*

Recall Baer's theorem that an abelian group is injective iff it is divisible.

Theorem 17.9. *$\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator.*

Proof. To show this we must show that if $\varphi : A \to B$ is a nonzero homomorphism of abelian groups, then there exists some $r : B \to \mathbb{Q}/\mathbb{Z}$ such that $r \circ \varphi = 0$. Since $\varphi \neq 0$, there is some a such that $b = \varphi(a) \neq 0$ and we can restrict our attention to b, so WLOG $A = \mathbb{Z}$ (and more generally we can restrict our attention to morphisms from generators). The subgroup $\langle b \rangle \subset B$ admits a nonzero morphism into $\mathbb{Q}/\mathbb{Z}$ which extends to B by injectivity. □

Exercise 17.10. *For abelian groups there is a canonical short exact sequence $A/\,Tors(A) \leftarrow A \leftarrow Tors(A)$. Find a compatible sequence of injective resolutions of length 2, and conclude that the total right derived functor of $\otimes : Ab \times Ab \to Ab$ is $A \otimes B \otimes \mathbb{Q}$ in degree 0 and 0 otherwise.*

On the other hand, we can also take right derived functors of the functor $-\otimes B$. This gives functors $(R^qF)(A)$ given by the cohomology of an injective resolution of A tensored with B. Let $I^0 \to I^1$ be such an injective resolution. Considering also the sequence $0 \leftarrow B/\mathrm{Tors}(B) \leftarrow B \leftarrow \mathrm{Tors}(B) \leftarrow 0$, we get an exact sequence

$$0 \leftarrow I^q \otimes (B/\mathrm{Tors}(B)) \leftarrow I^q \otimes B \leftarrow I^q \otimes \mathrm{Tors}(B) \leftarrow \mathrm{Tor}^1_{\mathbb{Z}}(I^q, B/\mathrm{Tors}(B)) \leftarrow \mathrm{Tor}^1_{\mathbb{Z}}(I^q, B) \tag{137}$$

for $q = 0, 1$. This implies that we can compute the right derived functors by taking the homology of the complex

$$0 \to A \otimes B/\mathrm{Tors}(B) \to I^0 \otimes B/\mathrm{Tors}(B) \to I^1 \otimes B/\mathrm{Tors}(B) \to 0 \tag{138}$$

which gives $(R^0(-\otimes B))(A) \cong A \otimes B/\mathrm{Tors}(B)$.

18 Comments on derived functors

Let A, B be abelian categories and $F : A \to B$ be an additive functor. The assignment $F \mapsto L_qF$ is itself a functor from the functor category $A \Rightarrow B$ to itself.

Exercise 18.1. *For any additive functor, L_qF is right exact and R^qF is left exact.*

There is a natural transformation $L_0F \to F$. This is the counit of an adjunction between functors and right exact functors as follows: applying L_0 to both sides gives $L_0L_0F \to F$, but since L_0F is right exact, $L_0L_0F \cong F$. More generally, if $\varphi : G \to F$ is a natural transformation where G is right exact, then the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & F \\ L_0(G) \arrow[u] \arrow[r] & L_0(F) \arrow[u] \end{tikzcd} \tag{139}$$

commutes; moreover, the map $L_0(G) \to G$ is an isomorphism. Hence G factors through L_0 as desired. In other words, the right exact functors $A \to B$ are a coreflective subcategory of the additive functors, and the right adjoint to the inclusion is L_0.

What is $L_q(L_pF)$? We compute this by applying L_pF to a projective resolution. However, L_pF vanishes on projective objects for $p \geq 1$, hence $L_q(L_pF) = 0$ for $p \geq 1$

and for all q.

What is $L_q(L_0F)$? We compute this by applying L_0F to a projective resolution. On projective objects, the natural map $L_0F \to F$ is an isomorphism, so $L_q(L_0F) \cong L_qF$.

A more interesting question is to compute $L_p(R^qF)$.

19 Loop and suspension

Associated to any abelian category A is a pair of homotopy categories, the projective homotopy category A/P and the injective homotopy category A/I. In the first category we identify arrows whose difference factors through a projective object, and in the second category we identify arrows whose difference factors through an injective object. Choosing for each $M \in A$ an epimorphism $P \to M$, and call its kernel LM. This assignment extends to a unique functor Ω (loop space) from the projective homotopy category of A to itself. Dually, we get a suspension functor Σ by considering the injective homotopy category.

Ω can be obtained from any functor $P(M)$ from A to the category of epimorphisms in A with projective source and target M. Dually, Σ can be obtained from any functor $I(M)$ from A to the category of monomorphisms in A with injective target and source M.

Given two objects M, N, choose sequences $M \leftarrow P_M \leftarrow \Omega M$ and $\Sigma N \leftarrow I^N \leftarrow N$. Then the diagram

$$\begin{tikzcd} M \arrow[d, "f"] & P_M \arrow[l] & \Omega M \arrow[l] \\ \Sigma N & I^N \arrow[l] & N \arrow[l] \end{tikzcd} \tag{140}$$

extends by the fundamental lemmas to a chain map, giving a map $\Omega f : \Omega M \to N$. Dually, the diagram

$$\begin{tikzcd} M & P_M \arrow[l] & \Omega M \arrow[l] \arrow[d, "g"] \\ \Sigma N & I^N \arrow[l] & N \arrow[l] \end{tikzcd} \tag{141}$$

extends by the fundamental lemmas to a chain map, giving a map $\Sigma g : M \to \Sigma N$.

This gives a diagram of homs

$$\begin{tikzcd}
\mathrm{Hom}_{A/P}(M,\Sigma N) \arrow[r] & \mathrm{Hom}_{A/P}(\Omega M, N) \\
\mathrm{Hom}_A(M,\Sigma N) \arrow[u] \arrow[d] & \mathrm{Hom}_A(M,\Omega N) \arrow[u] \arrow[d] \\
\mathrm{Hom}_{A/I}(M,\Sigma N) & \mathrm{Hom}_{A/I}(M,\Omega N) \arrow[l]
\end{tikzcd} \tag{142}$$

and if the projectives and injectives in A coincide, we can conclude that Ω and Σ are adjoint functors. This is true in particular for Frobenius categories, which are abelian categories with enough projectives and injectives such that the projectives and injectives coincide.

The topological analogue here is a pair of functors Σ, Ω on the homotopy category hTop_* of pointed topological spaces with Σ left adjoint to Ω. This is a special case of the tensor-hom adjunction, where we tensor and hom with the circle.

In any category there is an adjunction between classes of epimorphisms and classes of objects as follows: to a class of epimorphisms we associate the objects which have the lifting property characterizing projective objects, but only with respect to that class, and to a class of objects we associate the epimorphisms against which they all live. We can generalize resolutions to these classes.

20 Relative homological algebra

Given a homomorphism $\varphi : R \to S$ of unital rings, there is a restriction functor

$$S\text{-Mod} \to R\text{-Mod} \tag{143}$$

given by precomposing with φ and an induction functor

$$R\text{-Mod} \to S\text{-Mod} \tag{144}$$

given by tensoring which is its left adjoint. This is a special case of the tensor-hom adjunction. For example, if $R = k[H], S = k[G]$ where H is a subgroup of G and k is a commutative ring, with $\varphi : k[H] \to k[G]$ the obvious inclusion, resriction as defined above is the usual restriction of representations and its left adjoint is induction of representations.

Call an exact sequence of S-modules admissible or relatively acyclic if it splits as an exact sequence of R-modules. This notion of splitting induces a notion of relatively projective S-modules. S-Mod has enough relative projectives: in fact, any S-module induced from an R-module is relatively projective, so the counit

$$M \leftarrow S \otimes_R M \tag{145}$$

is a relative projective covering M. Iterating this construction gives the bar resolution. There is a dual notion of cobar resolution. Eilenberg-Moore has details. They work in a very general setting. All the fundamental lemmas work in this setting; we have a more restricted notion of acyclicity but a correspondingly less restricted notion of projectivity. This induces a notion of relative derived functors.

This is important in Hochschild (co)homology, which should be defined as the relative derived functors of tensor and hom with A as an $A \otimes_k A^{op}$-module (relative tor and ext). These are denoted

$$\mathrm{Tor}_q^{(A \otimes_k A^{op}, k)}(A, M) = H_q(A; M) \tag{146}$$

and

$$\mathrm{Ext}^q_{(A \otimes_k A^{op}, k)}(A, M) = H^q(A; M). \tag{147}$$

Some people use Hochschild cohomology to refer exclusively to $H^q(A; A)$. This controls infinitesimal deformations of A and has a Gerstenhaber structure given by the Schouten-Nijenhuis bracket. Hochschild cohomology $HH^q(A)$ should properly speaking refer to $H^q(A; A^*)$ where A^* is the linear dual of A, whereas Hochschild homology $HH_q(A)$ should refer to $H_q(A; A)$.

Hochschild homology and cohomology reduces in special cases to group and Lie algebra homology and cohomology (when we take A to be a group algebra resp. a universal enveloping algebra). However, the standard complex used to compute Lie algebra (co)homology is not the bar resolution; rather, it is the Chevalley-Eilenberg complex.

If R is a k-algebra, for any $x \in R$ we may define a chain complex

$$0 \to R \xrightarrow{x} R \to 0. \tag{148}$$

Tensoring n versions of this complex over k associated to elements $x_1, ...x_n$ gives the Koszul complex for R with respect to $x_1, ...x_n$. When $R = k[x_1, ...x_n]$, this gives a free resolution of k as an R-module. Writing $R = S(V^*)$ where V is a free finite k-module, the Koszul complex can be written in terms of tensor products of symmetric and exterior algebras. The entire complex may be thought of as a symmetric algebra on a supermodule. When $R = U(\mathfrak{g})$ is a universal enveloping algebra, we may take $x_1, ...x_n$ to be a basis of $\mathfrak{g}$ and we will obtain the Chevalley-Eilenberg resolution.

21 Calculus of (right) fractions

Let C be a category and let Σ be a subcategory. We would like to invert the morphisms in Σ; furthermore, we would like to represent morphisms in the resulting category using right roofs

$$b \overset{\beta}{\leftarrow} x \overset{\sigma}{\rightarrow} a \tag{149}$$

where $\sigma \in \Sigma$; we want to think of this morphism as a right fraction $b \xleftarrow{\beta \circ \sigma^{-1}} a$. This cannot be done in general (in general we need zigzags of morphisms), but it can be done if Σ satisfies certain conditions, the Ore conditions.

The first condition is that any left roof

$$b \overset{\sigma}{\rightarrow} x \overset{\alpha}{\leftarrow} a \tag{150}$$

can be replaced by a right roof in the sense that there exists a commutative diagram

$$\begin{tikzcd} x' \arrow[r, "\sigma'"] \arrow[d, "\alpha'"'] & a \arrow[d, "\alpha"] \\ b \arrow[r, "\sigma"'] & x \end{tikzcd} . \tag{151}$$

Here $\sigma, \tau \in \Sigma$ and $\alpha, \beta, \gamma \in C$. This condition is necessary to make sense of composition.

The second condition is that if a parallel pair of morphisms α, β is coequalized by a morphism from Σ, then it is equalized by a morphism from Σ. This is necessary for a certain relation to be an equivalence relation.

For any two objects a, b in C, consider the category of right roofs from a to b. The objects are right roofs and the morphisms are commutative diagrams. Consider the connected components π_0 of this category (the equivalence classes of the equivalence relation generated by the existence of a morphism between two objects). We would like to define

$$\mathrm{Hom}_{C[\Sigma^{-1}]}(a, b) = \pi_0 \left(\text{roofs } a \to b\right) \tag{152}$$

but there are size issues to doing this.

Proposition 21.1. *Two right roofs are in the same connected component if and only if they both admit a morphism to the same right roof.*

Example Let C be a category of chain complexes in an abelian category A and let Σ be the subcategory of quasi-isomorphisms. In general, Σ does not satisfy the Ore conditions. However, if C is the homotopy category of chain complexes in A, then the quasi-isomorphisms satisfy the Ore conditions. However, in general there are still size issues.

The first Ore condition would be satisfied if the pullback of a quasi-isomorphism is always a quasi-isomorphism. This is not always true; however, the pullback of an epic quasi-isomorphism is a quasi-isomorphism, and up to homotopy any quasi-ismorphism is an epic quasi-isomorphism.

A category is well-powered if the category of subobjects of any object is essentially small. Gabber showed that if A is a well-powered abelian category with enough projectives, and moreover if all filtered colimits exist in A and are exact, then the localization $D(A)$ of the homotopy category $K(A)$ by quasi-isomorphisms exists and has a calculus of right fractions.

The localization $D^+(A)$ (the derived category) of $K^+(A)$ exists and has a calculus of right fractions as long as A has enough projectives and the inclusion $K^+_{\text{cofib}}(A) \subset K^+(A)$ (the subcategory of cofibrant objects, namely those with projective terms) is an equivalence.

Theorem 21.2. *A quasi-isomorphism between two cofibrant complexes in $C^+(A)$ is a chain homotopy equivalence.*

Derived categories are a natural setting for studying derived functors. For example, $\mathrm{Hom}_{D^+(A)}(M, N[q]) = \mathrm{Ext}^q(M, N)$. More generally, if $F : A \to B$ is an additive functor, we may consider the induced functor $K^+_{\text{cofib}}(A) \to K^+(B)$.