208 C*-algebras

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Recommended text: Davidson, C*-algebras

1 Introduction

The seeds of this subject go back to von Neumann, Heisenberg, and Schrodinger in the 1920s; observables in quantum mechanics should correspond to self-adjoint operators on Hilbert spaces, and the abstract context for understanding self-adjoint operators is C*-algebras. In the 1930s, von Neumann wrote about what are now called von Neumann algebras, namely subalgebras of the algebra of operators on a Hilbert space closed under adjoints and in the strong operator topology. This subject is sometimes called noncommutative measure theory because a commutative von Neumann algebra is isomorphic to $L^\infty(X)$ for some measure space X.

In 1943, Gelfand and Naimark introduced the notion of a C*-algebra, namely a Banach algebra with an involution * satisfying $\|a^*\| = \|a\|$ and $\|a^*a\| = \|a\|^2$. They showed that if such an algebra A is commutative, then it is isomorphic to the C*-algebra $C(X)$ of continuous complex-valued functions on a compact Hausdorff space X. This space X is obtained as the Gelfand spectrum of unital C*-algebra homomorphisms $A \to \mathbb{C}$.

Noncommutative examples include the algebra $B(H)$ of bounded operators on a Hilbert space. Gelfand and Naimark also showed that any C*-algebra is *-isomorphic to a *-algebra of operators on a Hilbert space. This subject is sometimes called noncommutative topology (as C*-algebras behave like the algebra of functions on a compact Hausdorff space).

From the 1960s to the 1980s, a new emphasis in the subject was on noncommutative algebraic topology (e.g. K-theory and K-homology). From the 1980s on, Connes advanced a program of noncommutative differential geometry (cyclic homology as an analogue of de Rham cohomology), in particular with noncommutative analogues of Riemannian metrics (Dirac operators). In 1994 Connes wrote a book (Noncommutative Geometry) which is out of print but available online.

If G is a locally compact (Hausdorff) topological group, then we can construct several important C*-algebras such as $C^*(G)$ from it related to the representation theory of G. If A is a C*-algebra and G a locally compact group acting on A, then we can define a crossed product C*-algebra $A \rtimes G$. There is an analogous construction for foliated manifolds. Another quite different example is the CAR-algebra.

2 Basics

We will be using quantum tori as examples throughout the course. These are in some sense the simplest nontrivial example of noncommutative differentiable manifolds.

2.1 Spectra

Let A be a unital C*-algebra.

Definition The *spectrum* $\sigma(a)$ of an element $a \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible. The *spectral radius* $\nu(a)$ is the supremum of the absolute values of the elements of $\sigma(a)$.

If A is commutative, then $A \cong C(X)$ for some compact Hausdorff space X, the spectrum is the range, and the spectral radius is the supremum norm. Furthermore, if $a \in A$ is self-adjoint, then $\|a\| = \nu(a)$ by spectral permanence (below). This implies that $\|a\|^2 = \|a^*a\| = \nu(a^*a)$, hence the norm on a C*-algebra is completely determined by its *-algebra structure.

Lemma 2.1. *Let A be a unital C*-algebra and B a unital C*-subalgebra of A. Let $b \in B$. If b is invertible in A, then it is invertible in B.*

Proof. First assume b is self-adjoint. Let D be the C*-subalebra generated by $b, 1$. Then $D \cong C(X)$ for some X. If b is not invertible in B, then there exists $x \in X$ such that $b(x) = 0$ (where b is regarded as a function on X). It follows that for all $\epsilon > 0$ we can find $h \in B$ such that $\|h\| = 1$ and $\|bh\| < \epsilon$.

If a is an inverse to b in A, let $\epsilon = \frac{1}{2\|a\|}$. Then

$$1 = \|h\| = \|hba\| \leq \|hb\|\|a\| < \frac{1}{2} \tag{1}$$

which is a contradiction. For general b, we know that b^*b is self-adjoint. Hence if b is invertible in A, the above argument applies to b^*b, from which we conclude that b is invertible in B. □

Corollary 2.2. *(Spectral permanence) With hypotheses as above, $\sigma_B(b) = \sigma_A(b)$.*

2.2 Positivity

Definition Let A be a C*-algebra. An element $a \in A$ is *positive* if $a = a^*$ and $\sigma(a) \subset \mathbb{R}_{\geq 0}$.

Theorem 2.3. *The following are equivalent:*

1. *a is positive.*
2. *$a = b^2$ for some self-adjoint $b \in A$.*
3. *$a = c^*c$ for some $c \in A$.*

The positive elements of A form a cone A^+; that is, if a, b are positive, then $a + b$ is positive and ra is positive for all $r \in \mathbb{R}_{\geq 0}$. Furthermore, for any self-adjoint $a \in A$, we can write $a = a^+ - a^-$ where $a^+, a^- \in A^+$ and $a^+a^- = 0$, and $A^+ \cap (-A^+) = \{0\}$.

Proof. $1 \Leftrightarrow 2$: if a is positive, then the C*-subalgebra generated by a has the form $C(X)$ for some X. Then a is represented by a non-negative function which therefore has a square root. The converse is similar.

$2 \Rightarrow 3$: easy.

$3 \Rightarrow 2$: tricky, will do later. The last claim is straightforward, and the second-to-last claim will be left as an exercise. □

Exercise 2.4. *Verify that if $a \in A^+$ then $ra \in A^+$ for $r \in \mathbb{R}_{\geq 0}$ (easy) and that if a, b are positive then $a + b$ is positive (not easy).*

The solution to the hard exercise is as follows. The first proof was due to Fukumiya in 1952 and next in 1953 to John Kelley and Robert Vaught. The idea is that in $C(X)$ the relation $\geq$ (where we write $a \geq 0$ to mean that a is positive and we write $a \geq b$ to mean that $a - b \geq 0$) can be expressed in terms of the norm.

Lemma 2.5. *For self-adjoint $f \in C(X)$, the following are equivalent:*

1. *$f \geq 0$.*
2. *For all $t \geq \|f\|$, we have $\|f - t\| \leq t$.*
3. *For at least one $t \geq \|f\|$, we have $\|f - t\| \leq t$.*

Proof. 1 $\Rightarrow$ 2: for any $x \in X$ and for any $t \geq \|f\|$ we have $0 \geq f(x) - t \geq -t$, so $\|f - t\| \leq t$.

2 $\Rightarrow$ 3: obvious.

3 $\Rightarrow$ 1: if $t \geq \|f\|$ and $\|f - t\| \leq t$, then for any $x \in X$ we have $|f(x) - t| \leq \|f - t\| \leq t$, hence $f(x) \geq 0$. $\square$

Let a, b be two positive elements. Let $s = \|a\|$, so $\|a - s\| \leq s$. Let $t = \|b\|$, so $\|b - t\| \leq t$. Then

$$\|a + b\| \leq \|a\| + \|b\| = s + t \tag{2}$$

and

$$\|(a + b) - (s + t)\| \leq \|a - s\| + \|b - t\| \leq s + t. \tag{3}$$

Proposition 2.6. *A^+ is closed.*

Proof. Suppose a_λ is a net converging to a. Then in particular $\|a_\lambda\|$ converges to $\|a\|$. Choose t such that $t \geq \|a_\lambda\|$ for all λ. Then

$$\|a_\lambda - t\| let \tag{4}$$

hence

$$\|a - t\| = \lim \|a_\lambda - t\| \leq t \tag{5}$$

and the conclusion follows. $\square$

We now want to show that for any $c \in A$ we have $c^*c \geq 0$. Suppose otherwise. We know that the C*-subalgebra generated by c^*c has the form $C(X)$ for some X. Choose $b \geq 0$ in this subalgebra such that $c^*cb \neq 0$ and $-c^*cb^2 \geq 0$, which is possible because c^*c attains a negative value. Then

$$bc^*cb = (cb)^*(cb). \tag{6}$$

Let $d = cb$. Then $d \in A$ is nonzero and satisfies $-d^*d \in A^+$. Let $d = h + ik$ where h, k are self-adjoint. Then we compute that

$$d^*d + dd^* = 2h^2 + 2k^2 \in A^+ \tag{7}$$

from which it follows that

$$dd^* = 2(h^2 + k^2) - d^*d \geq 0. \tag{8}$$

Hence $\sigma_A(dd^*) \subset \mathbb{R}_{\geq 0}$ while $\sigma_A(d^*d) \subset \mathbb{R}_{\leq 0}$.

This will lead to a contradiction as follows.

Proposition 2.7. *If $a, b \in A$, then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

Proof. It suffices to show that $1 - ab$ is invertible if and only if $1 - ba$ is invertible. This follows formally from the identity

$$(1 - ab)^{-1} = 1 + a(1 - ba)^{-1}b \tag{9}$$

which one can guess by writing down a geometric series formally describing each side. □

Proposition 2.8. *Let A be a (unital) C*-algebra and $a, b \in A$ such that $0 \leq a \leq b$. Then $\|a\| \leq \|b\|$.*

Proof. Observe that $\|b\| - b \geq 0$. We also know that $b - a \geq 0$, from which it follows that $\|b\| - a \geq 0$ since the sum of two positive elements is positive, so $\|b\| \geq a \geq 0$. But $\|b\|$ and a commute, so the conclusion follows. □

Proposition 2.9. *If $0 \leq a \leq b$ and if $c \in A$, then $0 \leq c^*ac \leq c^*bc$.*

Proof. $a \leq b$ means $b - a \geq 0$, so $b - a = d^*d$. Thus $c^*(b - a)c = (dc)^*(dc) \geq 0$. □

2.3 Constructions

If I is a closed 2-sided ideal of a C*-algebra A, then the quotient A/I is also a C*-algebra.

The forgetful functor from unital Banach algebras to non-unital Banach algebras has a left adjoint which is obtained by adjoining an identity. The unitalization construction $A \mapsto \tilde{A}$ is given by pairs (α, a) where $\alpha \in \mathbb{C}$ and $a \in A$ which add in the obvious way and multiply so that $(1, 0)$ is the identity. The obvious norm here is $\|(\alpha, a)\| = \|\alpha\| + \|a\|$, and this gives a Banach algebra. However, if we start with a non-unital C*-algebra, we do not necessarily get a unital C*-algebra.

To fix this, let A be a C*-algebra. Consider A^n regarded as a right A-module. Define an A-valued inner product by

$$\langle (a_k), (b_k) \rangle_A = \sum_k a_k^* b_k. \tag{10}$$

The motivating example is that $A = C(X)$, where the inner product behaves like a metric on the trivial bundle $X \times \mathbb{C}^n$. From such an A-valued inner product we can induce a norm given by $|\langle v, v \rangle_A|^{1/2}$.

Now suppose A is non-unital and regard A as a right A-module. Given $(\alpha, a) \in \tilde{A}$ (the unitalization), we can define a module endomorphism given by $T_{(\alpha,a)} : b \mapsto \alpha b + ab$ and define

$$\|(\alpha, a)\| = \|T_{(\alpha,a)}\| \tag{11}$$

where the second is the operator norm. This is a C*-norm on $\tilde{A}$, which can be proven using basic properties of the A-valued inner product.

In some situations we might not want to adjoin an identity. We can sometimes avoid this using approximate identities.

Definition Let A be a normed non-unital algebra and $\{e_\lambda\}$ a net of elements of A. Such a net is a *left approximate identity* if $\lim e_\lambda a = a$ for all $a \in A$. An approximate identity is *bounded* if $\|e_\lambda\| \leq k$ for all λ, and is of norm 1 if $\|e_\lambda\| \leq 1$.

Similarly we have right and two-sided approximate identities.

Proposition 2.10. *Let A be a unital C*algebra and let L be a left ideal in A, not necessarily closed. Then L has a right approximate identity of norm 1 consisting of elements of A^+. If A is separable, then we can arrange for the approximate identity to be a sequence.*

Proof. Let S be a dense subset of L (countable if A is separable). Let Λ be the collection of all finite subsets of S ordered by inclusion. Given $\lambda \in \Lambda$, let

$$b_\lambda = \sum_{a \in \lambda} a^* a. \tag{12}$$

Since L is a left ideal, $b_\lambda \in L$ and is positive. Let

$$e_\lambda = \left(\frac{1}{|\lambda|} + b_\lambda\right)^{-1} b_\lambda. \tag{13}$$

Again we have $e_\lambda \in L$, and again e_λ is positive. The claim is that e_λ is a right approximate identity for L.

To see this, let $a \in S$. Consider $\|a - ae_\lambda\|$. If $a \in \lambda$, then

$$\|a - ae_\lambda\|^2 = \|a(1 - e_\lambda)\|^2 = \|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \tag{14}$$

but since $a \in \lambda$ we know that $a^*a \leq b_\lambda$, so we conclude by two previous inequalities that

$$\|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \leq \|(1 - e_\lambda)b_\lambda(1 - e_\lambda)\| = \|(1 - e_\lambda)^2 b_\lambda\|. \tag{15}$$

We now compute that

$$(1 - e_\lambda)^2 b_\lambda = \frac{1}{n}\frac{b_\lambda}{\frac{1}{n} + b_\lambda} \tag{16}$$

whose norm tends to 0 as desired (where $n = |\lambda|$). This establishes that e_λ is a right approximate identity for elements in S, and the general conclusion follows by boundedness and density. □

Proposition 2.11. *Let I be a closed 2-sided ideal of a unital C*-algebra A. Then I is closed under taking adjoints, so is a (non-unital) C*-algebra.*

Proof. Let e_λ be a right approximate identity for I satisfying the conditions above. Let $a \in I$. Then

$$\|a^* - e_\lambda a^*\| = \|a - ae_\lambda\| \to 0. \tag{17}$$

Since I is a two-sided ideal, $e_\lambda a^* \in I$, hence $a^* \in I$ by closure. □

Proposition 2.12. *Any C*-algebra A, not necessarily unital, contains a two-sided approximate identity of norm 1 consisting of elements of A^+.*

Proof. Since A is a closed left ideal, it has a right approximate identity e_λ. Since A is closed under taking adjoints, the right approximate identity is a two-sided approximate identity. □

Let I and J be two-sided closed ideals. Then $I \cap J \supseteq IJ$ (the closure of the span of products of elements of I and elements of J). Now take an approximate identity e_λ in I. Then if $a \in I \cap J$ we have $\|a - ae_\lambda\| \to 0$ where $ae_\lambda \in IJ$, hence $a \in IJ$, so $I \cap J = IJ$.

Also, if I is a two-sided closed ideal of A and J is a two-sided closed ideal of I, then in fact J is an ideal of A.

Let A be a non-unital Banach algebra and I a (proper) closed 2-sided ideal of it. Then A/I is a Banach space (since I is closed) and a Banach algebra (since I is a 2-sided ideal).

Theorem 2.13. *(Segal) If A is a C*-algebra, so is A/I.*

We will adjoin a unit as necessary. The key fact is that if $\{e_\lambda\}$ is a positive norm-1 identity for I, then $\|1 - e_\lambda\| \leq 1$, and is in fact equal to 1.

Proof. We will need the following lemma.

Lemma 2.14. *For all $a \in A$ we have $\|a\|_{A/I} = \lim \|a - ae_{|lambda}\|$.*

Proof. Fix $\epsilon > 0$. Choose $d \in I$ such that $\|a - d\| \leq \|a\|_{A/I} + \epsilon$. Then

$$\|a - ae_\lambda\| = \|a(1-e_\lambda)\| \leq \|(a-d)(1-e_\lambda)\| + \|d(1-e_\lambda)\| \leq \|a-d| + \|d - de_\lambda\| \quad (18)$$

which is less than or equal to $\|a\|_{A/I} + 2\epsilon$ for λ suitably far into the net. We used the key fact above. ☐

Now we want to show that the norm on A/I satisfies the C*-identity. We have

$$\|a\|^2_{A/I} = \lim \|a(1-e_\lambda)\|^2 = \lim \|(1-e_\lambda)a^*a(1-e_\lambda)\| \leq \limsup \|a^*a(1-e_\lambda)\| \leq \|\|a^*a\| \quad (19)$$

by the key lemma as desired. ☐

2.4 Representations

Definition For a *-algebra A, a **-representation* of A on a vector space H with pre-inner product is a map $\pi : A \to (B(H))$ such that $\langle \pi(a)v, w\rangle = \langle v, \pi(a^*)w\rangle$. The representation is *nondegenerate* if the span $\{\pi(a)v : a \in A, v \in H\}$ is dense in H.

How do we represent C*-algebras on Hilbert spaces? For those of the form $C(X)$ we can choose a nice positive measure on X and look at the multiplication action on $L^2(X)$. Radon would be nice (these are positive linear functionals on $C(X)$ by Riesz-Markov). Such a measure gives us a pre-inner product

$$\langle f, g \rangle = \int \bar{f} g \, d\mu = \mu(\bar{f} g) \tag{20}$$

which we can get a Hilbert space out of. So in general it seems like a good idea to find nice linear functionals.

Let A be a unital *-algebra. A linear functional μ on A is *positive* if $\mu(a^* a) \geq 0$ for all $a \in A$. It is a *state* if it is positive and $\mu(1) = 1$ (these should be thought of as noncommutative probability measures).

Example Let A be a *-algebra of operators on a Hilbert space H. Then if $\psi \in H$, the linear functional

$$a \mapsto \langle a\psi, \psi \rangle \tag{21}$$

is a state iff ψ is a unit vector in H. This is a pure state in quantum mechanics.

If A, μ are as above, $\langle a, b \rangle = \mu(a^* b)$ defines a pre-inner product on A. The vectors of length 0 form a subspace N by the Cauchy-Schwarz inequality. A/N therefore inherits an inner product which we can complete to obtain a Hilbert space $L^2(A, \mu)$. This is the Gelfand-Naimark-Segal construction.

A has a left regular representation $\pi_a(b) = ab$ on itself which descends to A/N because N is a left ideal. Moreover, this action is compatible with adjoints. However, π_a is not necessarily bounded in general, so this action does not necessarily extend to the completion in general.

We need to check that $\langle a, b \rangle$ is actually a pre-inner product. In particular, we need to check that

$$\langle b, a \rangle = \overline{\langle a, b \rangle}. \tag{22}$$

To see this, note that

$$0 \leq \langle a + b, a + b \rangle = \langle a, a \rangle + \langle a, b \rangle + \langle b, a \rangle + \langle b, b \rangle \tag{23}$$

from which it follows that the imaginary part of $\langle a, b\rangle + \langle b, a\rangle$ is zero. Substituting ib for b we also conclude that the imaginary part of $\langle a, ib\rangle + \langle ib, a\rangle$ is zero, and this gives the result.

The above is equivalent to the claim that μ is *-linear if A has an identity but not in general. As a counterexample, let A be the algebra of polynomials vanishing at 0 and let $\mu(p) = ip'(0)$. Then $\langle p, q\rangle = 0$ identically but μ is not *-linear.

Now let N be the subspace of vectors of norm zero. By Cauchy-Schwarz, N is also $\{a : \langle a, b\rangle = 0 \forall b \in A\}$ and hence is a subspace. Furthermore, if $b \in N, a \in A$, then

$$\langle ab, ab\rangle = \langle b, a^* ab\rangle = 0 \tag{24}$$

hence $ab \in N$ and N is a left ideal. It follows that A/N is a left A-module, and the pre-inner product descends to an inner product on A/N which we can complete. We would like to extend the action of A to an action on the completion, but we do not have boundedness in general. For example, if A is the space of polynomial functions on $\mathbb{R}$ and

$$\mu(p) = \int_{-\infty}^{\infty} p(t) e^{-t^2} \, dt \tag{25}$$

then we can check that the action of A is not bounded with respect to the corresponding inner product. So we need more hypotheses.

Theorem 2.15. *Let A be a *-Banach algebra with identity and let μ be a positive linear functional on A. Then μ is continuous and $\|\mu\| = \mu(1)$.*

Corollary 2.16. *Let A be a *-normed algebra with identity and let μ be a continuous positive linear functional on A. Then $\|\mu\| = \mu(1)$.*

Proof. We will need the following lemma.

Lemma 2.17. *Let $a \in A$ be self-adjoint with norm strictly less than 1. Then there exists a self-adjoint element b such that $1 - a = b^2$.*

To show this we can write $\sqrt{1-a}$ as a convergent power series.

It follows that $\mu(1-a) \geq 0$, hence that $\mu(1) \geq \mu(a)$ for a self-adjoint with norm strictly less than 1. Similarly, $\mu(1+a) \geq 0$, hence $\mu(1) \geq -\mu(a)$, so $\mu(1) \geq |\mu(a)|$.

It follows by a limiting argument that for any self-adjoint $a \in A$ we have $\mu(1)\|a\| \geq |\mu(a)|$. For arbitrary a,

$$|\mu(a)|^2 = |\langle 1, a\rangle|^2 \leq \langle 1, 1\rangle\langle a, a\rangle = \mu(1)\mu(a^*a) \leq \mu(1)^2\|a^*a\| \leq \mu(1)^2\|a\|^2 \tag{26}$$

and the conclusion follows. □

Proposition 2.18. *Let A be a *-normed algebra with identity and let μ be a continuous positive linear functional. Then the left regular representation of A on A/N with inner product $\langle\cdot,\cdot\rangle$ is by bounded operators and extends to the completion; moreover, $\|\pi_a\| \leq \|a\|$.*

Proof. Let $a, b \in A$. Define $\mu_b(c) = \mu(b^*cb)$. Then μ_b is also a continuous positive linear functional. Now:

$$\langle \pi_a(b), \pi_a(b)\rangle = \langle ab, ab\rangle = \mu(b^*a^*ab) = \mu_b(a^*a) \tag{27}$$

which is less than or equal to

$$\|\mu_b\|\|a^*a\| = \mu_b(1)\|a^*a\| = \mu(b^*b)\|a^*a\| \leq \|a\|^2\langle b, b\rangle \tag{28}$$

and the conclusion follows. □

So let A be a *-normed algebra with identity and μ be a continuous positive linear functional on A. Then $L^2(A, \mu)$ is a nondegenerate *-representation of A

Corollary 2.19. *For every continuous positive linear functional μ on a *-normed algebra A, there is a *-representation of A and a vector $v \in H$ such that $\mu(a) = \langle \pi(a)v, v\rangle$.*

Let (π, H) be a *-representation of A on a Hilbert space H and let v be a vector in H. Let

$$K_v = \overline{\{\pi(a)v : a \in A\}}. \tag{29}$$

Then K_v is invariant under the action of A; in fact it is the minimal closed subspace invariant under A containing v. We call K_v the *cyclic subspace* generated by v. The representations $L^2(A, \mu)$ is cyclic with cyclic vector the image of the identity.

Proposition 2.20. *Let* (π^1, H^1, v^1) *and* (π^2, H^2, v^2) *be cyclic *-representations of* A *with cyclic vectors* v^1, v^2. *Let* μ^1, μ^2 *be the corresponding states. If* $\mu^1 = \mu^2$ *then the representations are* unitarily equivalent.

Proof. We want to take $U(\pi^1(a)v^1) = \pi^2(a)v^2$. The problem is that this may not be well-defined. But we can compute that

$$\langle U(\pi^1(a)v^1), U(\pi^1(b), v^1)\rangle = \mu^2(b^*a) = \mu^1(b^*a) = \langle \pi^1(a)v^1, \pi^2(b)v^1\rangle \tag{30}$$

hence U is well-defined and unitary. The same argument works in the other direction. □

Hence there is a bijective correspondence between continuous positive linear functionals on A and unitary equivalence classes of (pointed) cyclic *-representations of A.

Let H_λ be a family of Hilbert spaces. We can form their Hilbert space direct sum $\bigoplus_{\lambda\in\Lambda} H_\lambda$, which is the subspace of the ordinary product where the inner product

$$\langle v, w\rangle = \sum_{\lambda\in\Lambda}\langle v_\lambda, w_\lambda\rangle \tag{31}$$

converges (equivalently the completion of the ordinary direct sum).

If for each λ we have a representation π_λ of A on H_λ, we can try to define a representation on the direct sum. This is possible if and only if $\|\pi_\lambda(a)\|$ is uniformly bounded in λ. In practice we will have $\|\pi_\lambda(a)\| \le \|a\|$, in which case the same will be true of the direct sum.

Proposition 2.21. *If* $K \subset H$ *is an invariant subspace of a *-representation, then so is its orthogonal complement* $K^\perp$.

Proof. If $w \in K^\perp$, then

$$\langle \pi(a)v, w\rangle = 0 \qquad \forall v \in K, a \in A \Leftrightarrow \tag{32}$$

$$\langle v, \pi(a)^*w\rangle = 0 \qquad \forall v \in K, a \in A \Leftrightarrow \tag{33}$$

$$\langle v, \pi(a^*)w\rangle = 0 \qquad \forall v \in K, a \in A \Leftrightarrow \tag{34}$$

$$\pi(a^*)w \in K^\perp \qquad \forall a \in A \tag{35}$$

as desired. ☐

Corollary 2.22. *Finite-dimensional *-representations are completely reducible (semisimple).*

Proposition 2.23. *Every *-representation on a Hilbert space H is a Hilbert space direct sum of cyclic representations.*

Proof. Imitate the proof that a Hilbert space has an orthonormal basis. ☐

(We don't get a decomposition into irreducible representations in general. We may need to use direct integrals instead of direct sums, and then direct integral decompositions need not be unique.)

Definition The *universal *-representation* of A is the Hilbert space direct sum of all $L^2(A, \mu)$ as μ runs over all positive linear functionals of norm 1 (states).

The universal *-representation contains every cyclic representation. It follows that every *-representation is a direct summand of some sum of copies of the universal *-representation.

A need not have any *-representations or states. For example, let $A = \mathbb{C} \times \mathbb{C}$ with the sup norm and $(\alpha, \beta)^* = (\bar{\beta}, \bar{\alpha})$.

Lemma 2.24. *Let A be a C^*-algebra and μ a continuous linear functional such that $\|\mu\| = \mu(1)$. Then μ is positive.*

Proof. We want to show that $\mu(a^* a) \geq 0$. Equivalently, we want to show that if $a \in A^+$ then $\mu(a) \geq 0$. Suppose otherwise. Let $B = C(X)$ be the *-algebra generated by a and restrict μ to B.

First, we want to show that if $f = \bar{f}$ (in $C(X)$) then $\mu(f) \in \mathbb{R}$. Suppose $\mu(f) = r + is$. Then for $t \in \mathbb{R}$ we have

$$|\mu(f + it)|^2 \leq \|f + it\|^2 = \|f\|^2 + t^2 \tag{36}$$

but the LHS is equal to

$$|r + i(s + t)|^2 = r^2 + s^2 + 2st + t^2 \tag{37}$$

and subtracting t^2 from both sides gives a contradiction unless $s = 0$.

Now, if $f \geq 0$ we have $\|f - \|f\|\| \leq \|f\|$, hence

$$|\mu(f) - \|f\|| = |\mu(f - \|f\|)| \leq \|f\| \tag{38}$$

which gives $\mu(f) \geq 0$ as desired. □

Theorem 2.25. *Let A be a unital C*-algebra and B a C*-algebra of A. Let μ be a positive linear functional on B. Then μ extends to a positive linear functional on A.*

Proof. Apply the Hahn-Banach theorem together with the lemma above. □

Theorem 2.26. *Let A be a unital C*-algebra and $a \in A$ self-adjoint. For any $\lambda \in \sigma(a)$, there is a cyclic nondegenerate *-representation (π, H, v) with $\|v\| = 1$ such that $\langle \pi(a)v, v\rangle = \lambda$.*

Proof. Let B be the C*-subalgebra generated by a, which is $C(\sigma(a))$. Let μ_0 be the functional defined on B given by the Dirac measure at λ. Then μ_0 is a state on B. Let $\tilde{\mu}_0$ be an extension of μ_0 to a state on A, and let (π, H, v) be the GNS representation associated to $\tilde{\mu}_0$. □

Corollary 2.27. *(Gelfand-Naimark) The universal representation of A is isometric (norm-preserving).*

Proof. Let $a \in A$. Then $\|\pi(a)\|^2 = \|\pi(a^*a)\|$. Let $\lambda = \|a^*a\|$. The corresponding representation π_λ satisfies $\|\pi_\lambda(a^*a)\| = \lambda = \|a^*a\|$, and the conclusion follows. □

Corollary 2.28. *All of the above continues to hold if A is a non-unital C*-algebra.*

Let A be a *-normed algebra with a norm-1 approximate identity.

Lemma 2.29. *Let μ be a continuous positive linear functional on A. Then*

1. $\mu(a^*) = \overline{\mu(a)}$.
2. $|\mu(a)|^2 \leq \|\mu\|\mu(a^*a)$.

Proof. Write

$$\mu(a^*) = \lim \mu(a^*e_\lambda) = \lim \langle a, e_\lambda\rangle_\mu \tag{39}$$

and since we know we have an inner product, this is equal to

$$\lim \overline{\langle e_\lambda, a\rangle} = \lim \overline{\mu(e_\lambda^* a)} = \overline{\mu(a)} \tag{40}$$

as desired. Next, write

$$|\mu(a)|^2 = \lim |\mu(e_\lambda a)|^2 = \lim |\langle e_\lambda^*, a\rangle_\mu|^2 \tag{41}$$

which by Cauchy-Schwarz is bounded by

$$\lim \langle a, a\rangle_\mu \langle e_\lambda^*, e_\lambda^*\rangle \le \mu(a^* a)\|\mu\| \tag{42}$$

as desired. □

Proposition 2.30. *Let A be a *-normed algebra and let μ be a continuous positive linear functional on A. If μ satisfies the properties above and if μ extends to the unitization $\tilde{A}$ of A so that $\tilde{\mu}(1) = \|\mu\|$, then $\tilde{\mu}$ is a positive linear functional.*

Proof. The nontrivial thing to prove is positivity. We compute that

$$\tilde{\mu}((a+\lambda 1)^*(a+\lambda 1)) = \mu(a^* a) + \bar{\lambda}\mu(a) + \lambda\overline{\mu(a)} + |\lambda|^2\|\mu\| \tag{43}$$

which is greater than or equal to

$$\mu(a^* a) - 2|\lambda||\mu(a)| + |\lambda|^2\|\mu\| \ge \mu(a^* a) - 2|\lambda|\|\mu\|^{1/2}\mu(a^* a)^{1/2} + |\lambda|^2\|\mu\| = \left(\mu(a^* a)^{1/2} - |\lambda|\|\mu\|^{1/2}\right)^2 \ge 0 \tag{44}$$

and the conclusion follows. □

Corollary 2.31. *Let A be a *-normed algebra with a norm-1 approximate identity. Let μ be a continuous positive linear functional on A. Extend it to $\tilde{A}$ by $\tilde{\mu}(1) = \|\mu\|$. Then $\tilde{\mu}$ is a positive linear functional on $\tilde{A}$.*

With the above hypotheses, we can apply the GNS construction to obtain a cyclic representation (π, H, v) of $\tilde{A}$.

Proposition 2.32. *If we restrict π to A, then v remains a cyclic vector for $\pi|_A$.*

Proof. Choose a self-adjoint sequence a_n with $\|a_n\| \le 1$ so that $\mu(a_n) \uparrow \|\mu\|$. Then

$$\|\pi(a_n)v - v\|^2 = \mu(a_n^* a_n) - \mu(a_n) - \mu(a_n^*) + \|\mu\| \le \|\mu\| - \mu(a_n) \to 0 \tag{45}$$

as desired. □

If A is a non-unital *-normed algebra, (π, H) a nondegenerate *-representation, and e_λ is an approximate identity of norm 1, then

$$\pi(e_\lambda)\pi(a)v = \pi(e_\lambda a)v \to \pi(a)v \tag{46}$$

hence in particular $\pi(e_\lambda)v \to v$ for all v in a dense subspace of H. Since e_λ has norm 1, it follows that $\pi(e_\lambda)v \to v$ for all $v \in H$. We conclude the following:

Proposition 2.33. *(π, H) is nondegenerate if and only if $\pi(e_\lambda)v \to v$ for all $v \in H$.*

Corollary 2.34. *Let A be a *-normed algebra with approximate identity e_λ. Let μ be apositive linear functional and (π, H, v) the corresponding GNS representation. Then $\mu(e_\lambda) \to \langle \pi(e_\lambda)v, v\rangle \to \langle v, v\rangle = \|\mu\|$.*

Let A be a unital *-normed algebra and $S(A)$ the space of positive linear functionals of norm 1 on A (states). $S(A)$ is a closed subset of the unit ball of the dual A^* in the weak-* topology, hence compact by Banach-Alaoglu. $S(A)$ is also convex, which suggests that it would be interesting to examine its extreme points.

The above fails if A is not unital. For example, if $A = C_0(\mathbb{R})$ is the space of real-valued functions on $\mathbb{R}$ vanishing at infinity, the Dirac deltas $\delta(n)$ have limit 0 as $n \to \infty$, so the space of states on A is not closed.

Schur's lemma in this context is the following.

Lemma 2.35. *(Schur) (π, H) is irreducible iff $End_A(H) \cong \mathbb{C}$.*

Note that $\mathrm{End}_A(H)$ is a C*-subalgebra of $B(H)$ closed in the strong operator topology (a von Neumann algebra).

Proof. If H is not irreducible, it has a proper invariant subspace K with invariant orthogonal complement $K^\perp$. It follows that the projection onto K belongs to $\mathrm{End}_A(H)$, which therefore cannot be isomorphic to $\mathbb{C}$.

If $\mathrm{End}_A(H)$ is not isomorphic to $\mathbb{C}$, then it is a C*-algebra containing an element T which is not a scalar multiple of the identity. The C*-subalgebra generated by T contains a zero divisor whose kernel is an invariant subspace of H, so H is not irreducible. $\square$

2.5 States

Definition Let μ, ν be positive linear functionals. Then $\mu \geq \nu$ if $\mu - \nu \geq 0$; we say that ν is *subordinate* to μ.

Given μ, let (π, H, v) be the corresponding GNS representation. Consider $\mathrm{End}_A(H)$. Let $T \in \mathrm{End}_A(H)$ be a positive operator smaller than the identity and set

$$\nu(a) = \nu_T(a) = \langle \pi(a)Tv, v \rangle. \tag{47}$$

(Then T is the Radon-Nikodym derivative of ν with respect to μ.) We compute that

$$\nu(a^*a) = \langle \pi(a^*a)Tv, v \rangle = \langle T^{1/2}\pi(a)v, T^{1/2}\pi(a)v \rangle \geq 0 \tag{48}$$

so ν is positive. Similarly, $\mu - \nu$ is positive. Moreover, if $\nu_T = \nu_S$ then $T = S$ (by nondegeneracy).

Conversely, suppose ν is a positive linear functional with $\mu \geq \nu \geq 0$, we want to show that $\nu = \nu_T$ for some $T \in \mathrm{End}_A(H)$. For $a, b \in A$ we have

$$|\nu(b^*a)| \leq \nu(a^*a)^{1/2}\nu(b^*b)^{1/2} \leq \mu(a^*a)^{1/2}\mu(b^*b)^{1/2} = \|\pi(a)v\|\|\pi(b)v\|. \tag{49}$$

In particular, if the RHS is zero, so is the LHS. For fixed b, we can attempt to write down a map

$$\pi(a)v \mapsto \nu(b^*a) \tag{50}$$

which is well-defined by the above inequality. This is a linear functional of norm less than or equal to $\|\pi(b)v\|$, so by the Riesz representation theorem there exists $T \in \mathrm{End}(H)$ such that

$$\nu(b^*a) = \langle \pi(a)v, T^*\pi(b)v \rangle. \tag{51}$$

Since $\nu \geq 0$ we have $T \geq 0$. Since $\nu \leq \mu$, we have $T \leq I$. To show that

$T \in \mathrm{End}_A(H)$, write

$$\begin{aligned} \langle \pi(c)T\pi(a)v, \pi(b)v\rangle &= \langle T\pi(a)v, \pi(c^*b)v\rangle && (52) \\ &= \nu((c^*b)^*a) && (53) \\ &= \nu(b^*ca) && (54) \\ &= \langle \pi(c)\pi(a)v, T^*\pi(b)v\rangle && (55) \end{aligned}$$

which gives $\pi(c)T = T\pi(c)$ by density. We can encapsulate our work above in the following.

Proposition 2.36. *The map $T \mapsto \nu_T$ is a bijection between $\{T \in End_A(H) : 0 \leq T \leq I\}$ and $\{\nu : \mu \geq \nu \geq 0\}$.*

Definition A positive linear functional is *pure* if whenever $\mu \geq \nu \geq 0$ then $\nu = r\mu$ for some $r \in [0, 1]$.

Theorem 2.37. *Let μ be a positive linear functional. Then μ is pure iff the associated GNS representation (π, H, v) is irreducible.*

Proof. If μ is not pure, there is $\mu \geq \nu \geq 0$ such that ν is not a multiple of μ. Then there exists $T \in \mathrm{End}_A(H)$ not a scalar multiple of the identity, so by Schur's lemma, H is not irreducible.

Conversely, if H has a proper invariant subspace, let $P \in \mathrm{End}_A(H)$ be the corresponding projection. Then ν_P is not a multiple of μ. □

Proposition 2.38. *(Recall that the space of states $S(A)$ is convex.) Let $\mu \in S(A)$. Then μ is pure if and only if μ is an extreme point of $S(A)$.*

Proof. If μ is not extreme, $\mu = t\mu_1 + (1 - t)\mu_2$ where $\mu_i \in S(A), t \in (0, 1)$, and $\mu_1 \neq \mu \neq \mu_2$. Then $\mu \geq t\mu_1$, so μ is not pure.

Conversely, suppose μ is not pure, so there is $\mu > \nu > 0$ with ν not a scalar multiple of μ. Then $\mu = (\mu - \nu) + \nu$ which is a convex linear combination of states (once the terms have been normalized) using the fact that $\|\mu - \nu\| = \lim(\mu - \nu)(e_\lambda)$. □

Extreme points of $S(A)$ are *pure states*. If A is unital, then $S(A)$ is weak-* compact, so by the Krein-Milman theorem has many extreme points. A sketch of Krein-Milman is as follows.

Definition Let K be a convex subset of a vector space. A *face* of K s a subset F such that if $v \in F$ and $v = tv_1 + (1-t)v_2$ with $0 < t < 1$ and $v_1, v_2 \in K$, then $v_1, v_2 \in F$.

A face of a face is a face.

Let V be a locally convex topological vector space. If K is a convex compact subset of V and if $\varphi \in V^*$ takes its maximum on K at some point v_0, then $\{v \in K : \varphi(v) = \varphi(v_0)\}$ is a closed face of K. By Hahn-Banach together with a Zorn's lemma argument we can find descending chains of closed faces which necessarily end at extreme points.

Let (π^a, H^a) be the direct sum of the GNS representations associated to all of the extreme points of $S(A)$.

Theorem 2.39. *For any $a \in A$ we have $\|\pi^a(a)\| = \|\pi^U(a)\| = \|a\|$ (where π^U is the universal representation).*

Proof. Let $a \in A$ be self-adjoint. Then there exists $\rho \in S(A)$ with $|\rho(a)| = \|a\|$. Let $S_e(A)$ be the set of extreme (pure) states of A. Suppose that there exists c such that $|\mu(a)| \leq c < \|a\|$ for all $\mu \in S_e(A)$. This inequality is preserved by convex combinations, so this inequality holds for any μ in the closure of the convex hull of $S_e(A)$, which by Krein-Milman is all of $S(A)$; contradiction. Hence there is a sequence $\mu_n \in S_e(A)$ such that $|\mu_n(a)| \uparrow \|a\|$. $\square$

As a corollary, Gelfand and Raikov showed that locally compact groups have many irreducible unitary representations.

Let A be a non-unital C*-algebra. We can define a quasi-state space $Q(A)$ to be the space of all positive linear functionals with $\|\mu\| \leq 1$. This is also a compact convex subset of A^* in the weak-* topology and its extreme points are the extreme states $S_e(A)$ together with 0.

2.6 Compact operators

Let H be a Hilbert space and let $B_f(H)$ be the set of finite rank operators in $B(H)$. This is a two-sided ideal in $B(H)$. If I is any proper 2-sided ideal, then $B_f(H) \subseteq I$. We define the *compact operators* $B_0(H)$ to be the closure of $B_f(H)$. Then $B_f(H)$ is the minimal dense ideal in $B_0(H)$. (The chain of inclusions $B_f(H) \subset B_0(H) \subset B(H)$ is analogous to the chain of inclusions $C_c(S) \subset C_0(S) \subset \ell^\infty(S)$ where S is a set.)

The identity operator is not compact, so $B_0(H)$ is a natural example of a non-unital C*-algebra. $B_0(H)$ is topologically simple in the sense that it has no proper 2-sided closed ideals. Its representation on H is irreducible. Moreover, every irreducible representation of $B_0(H)$ is unitarily equivalent to H, and every nondegenerate representation is a direct sum of copies of H.

Recall that two rings R, S are Morita equivalent if their categories of left modules are equivalent. R and $M_n(R)$ are known to be Morita equivalent. In some C*-algebraic sense, $B_0(H)$ is Morita equivalent to $\mathbb{C}$.

Theorem 2.40. *(Morita) Let $F : R\text{-Mod} \to S\text{-Mod}$ be an equivalence of categories. Then F is naturally isomorphic to tensoring by ${}_R X_S$ for some bimodule X.*

We return now to $B_0(H)$. For $v, w \in H$ define $\langle v, w\rangle_0 \in B_0(H)$ by $\langle v, w\rangle u = v\langle w, u\rangle$. This is a rank- 1 operator. If $T \in B(H)$ then $T\langle v, w\rangle_0 = \langle Tv, w\rangle_0$, hence $\langle v, Tw\rangle_0 = \langle v, w\rangle_0 T^*$.

Theorem 2.41. *Every non-degenerate representation of $B_0(H)$ is unitarily equivalent to a direct sum of copies of H.*

Proof. Let (π, V) be such a representation. For $T \in B_0(H), v \in V$ write $\pi(T)v$ as Tv. Pick any $h \in H, \|h\| = 1$ so that $\langle h, h\rangle_0$ is the projection onto h. Then $\langle h, h\rangle_0$ acts as a nonzero projection on V (because $B_0(H)$ is topologically simple). Let $v \in V, \|v\| = 1$ be such that $\langle h, h\rangle_0 v = v$. Define

$$Q : H \ni w \mapsto \langle w, h\rangle_0 v \in V. \tag{56}$$

A tedious computation shows that Q is an isometry and another computation shows that Q is a module homomorphism. The rest follows by Zorn's lemma. □

Corollary 2.42. *Every pure state of $B_0(H)$ is of the form $\mu_v(T) = \langle Tv, v\rangle$ for some $v \in H, \|v\| = 1$.*

Hence there is a bijection between the space of pure states on $B_0(H)$ and the projective space $\mathbb{P}(H)$. If P is a rank one operator, then we can also write $\mu_P(T) = \mathrm{tr}(TP)$. We can identify the space of states $S(B_0(H))$ with the set of positive trace class operators of trace less than or equal to 1 (density operators) (sometimes called mixed states).

Theorem 2.43. *(Burnside) Let A be a C*-subalgebra of $B_0(H)$ and assume that the standard representation of A on H is irreducible. Then $A = B_0(H)$.*

Proof. Let $T \in A$ be nonzero and positive. By the spectral theorem, $\sigma(T)$ is a countable set with 0 as its only possible accumulation point, and for any $r > 0$ the intersection $\sigma(T) \cap [r, \infty)$ is finite (or otherwise we could find a subspace on which T acts invertibly, which contradicts that T is approximable by finite-rank operators). The C*-subalgebra generated by T is isomorphic to $C(\sigma(T))$. The indicator function of any positive eigenvalue is approximable by polynomials in T, from which it follows that A contains nonzero projections. These projections necessarily have finite rank.

Let P be a projection in A of minimal rank. Then for any self-adjoint $S \in A$ we have $PSP \in A$. PSP acts on the range of P and has spectral projections of rank at most that of P. By minimality, PSP must be a scalar multiple $\alpha(S)P$ of P.

Suppose that v, w are orthogonal vectors in the range of P. Then for any $S \in A$ we have

$$\langle Sv, w\rangle = \langle SPv, Pw\rangle = \langle PSPv, w\rangle = \alpha(S)\langle v, w\rangle = 0. \tag{57}$$

Since any nonzero vector is cyclic, it follows that $w = 0$. Hence P is a rank 1 projection, so $P = \langle v, v\rangle_0$. Now for any $S \in A$ we have $SP \in A$ where $SP = \langle Sv, v\rangle_0$. Since A is norm-closed, $\langle v, v\rangle_0 \in A$ for all $v \in H$, so all rank-1 operators are in A, and the conclusion follows. □

Let A be a *-normed algebra and I be a 2-sided ideal in A with a bounded 2-sided approximate identity e_λ.

Proposition 2.44. *Let (π, H) be an irreducible representation of A. Then either $\pi(I) = 0$ or $\pi|_I$ is irreducible.*

Proof. Suppose $\pi(I) \neq 0$. Then $\overline{\pi(I)H}$ is an A-invariant subspace, hence is all of H, so the representation of I is nondegenerate.

Suppose K is a nontrivial I-invariant subspace. Then for $a \in A, i \in I, v \in K$ we have

$$\pi(a)\pi(i)v = \pi(ai)v \in K \tag{58}$$

and

$$\pi(a)v = \lim \pi(a)\pi(e_\lambda)v = \lim \pi(ae_\lambda)v \in K \tag{59}$$

from which it follows that K is A-invariant. $\square$

Proposition 2.45. *Let $(\pi, H), (\rho, K)$ be irreducible representations of A with $\pi(I) \neq 0$. Suppose that $\pi|_I, \rho|_I$ are unitarily equivalent. Then π, ρ are unitarily equivalent.*

Proof. Let $U : H \to K$ be a unitary equivalence. Then for any $a \in A, i \in I, v \in H$,

$$U(\pi(a)\pi(i)v) = U(\pi(ai)v) = \rho(ai)Uv = \rho(a)\rho(i)Uv = \rho(a)U\pi(i)v \tag{60}$$

and taking $i = e_\lambda$ and taking limits, the conclusion follows. $\square$

Proposition 2.46. *Let (π, H) be a nondegenerate representation of I. Then π has a unique extension $\tilde{\pi}$ to a representation of A on H.*

Proof. Suppose $\tilde{\pi}$ is such an extension. Then $\tilde{\pi}(a)\pi(i)v = \pi(ai)v$, and vectors of the form $\pi(i)v$ are dense, so $\tilde{\pi}$ is unique. To show existence, we want to define $\tilde{\pi}$ using

$$\tilde{\pi}(a)(\pi(i)v) = \pi(ai)v \tag{61}$$

and we want to show that this is well-defined. Hence we want to show that if $\sum \pi(i_j)v_j = 0$ then $\sum \pi(ai_j)v_j = 0$. Now

$$\sum \pi(ai_j)v_j = \lim \sum \pi(ae_\lambda i_j)v_j = \lim \pi(ae_\lambda) \sum \pi(i_j)v_j \tag{62}$$

and the conclusion follows modulo some technical details. $\square$

Theorem 2.47. *Let (π, H) be an irreducible representation of a C^*-algebra A. If $\pi(A) \cap B_0(H)$ is nonzero, then $B_0(H) \subseteq \pi(A)$. In this case, if $I = ker(\pi)$, then any two irreducible representations of A with kernel I are unitarily equivalent.*

Proof. Let $J = \pi^{-1}(B_0(H))$. Then J is a closed 2-sided ideal of A contained in I. Now, $\pi(J)$ is a closed *-subalgebra of $B_0(H)$. Since J is an ideal, $\pi|_J$ is irreducible, so by a previous result, $\pi(J) = B_0(H)$.

Now if (π, H) and (ρ, K) are two irreducible representations with kernel I, then $\pi|_J$ and $\rho|_J$ give irreducible representations of J, hence $J/I \cong B_0(H)$. So $\pi|_J$ and $\rho|_J$ are unitarily equivalent by a previous result, from which it follows that π, ρ are unitarily equivalent by a previous result. $\square$

Definition Let A be a C*-algebra. A is *CCR* (Kaplansky) or *liminal* (Dixmier) if in every irreducible representation (π, H) we have $\pi(A) = B_0(H)$. A is *GCR* (Kaplansky) or *post-liminal* (Dixmier) if $\pi(A) \supseteq B_0(H)$ for all π. Otherwise, A is *NGR* (Kaplansky) or *anti-liminal* (Dixmier).

Definition Let A be a C*-algebra. A *primitive ideal* is the kernel of an irreducible representation of A.

For a commutative C*-algebra, these are precisely the maximal ideals. If $\hat{A}$ is the space of primitive ideals and A is GCR, then there is a bijection between the space of unitary equivalence classes of irreducible representations and $\hat{A}$.

Theorem 2.48. *Let (π, H) be an irreducible representation of a C*-algebra A and let $I = ker(\pi)$. If $\pi(A) \cap B_0(H) = \{0\}$, then there are an uncountable number of equivalence classes of irreducible representations of A with kernel I. Furthermore, the set of all such representations is unclassifiable in a suitable sense.*

Suppose A is a unital, infinite-dimensional C*-algebra that is simple (has no proper 2-sided ideals). Then A is NGR and $\hat{A} = \{\{0\}\}$. In this situation it is hopeless to classify irreducible representations.

If G is a locally compact group, then one can construct various C*-algebras from it. If G is a connected Lie group that is

1. semisimple (e.g. $\mathrm{SL}_n(\mathbb{R}), \mathcal{O}(p,q)$), then the corresponding C*-algebras are CCR (Harish-Chandra).

2. nilpotent, then the corresponding C*-algebras are CCR (Dixmier).

3. solvable, then some are GCR and some are NGR.

For an NGR example, consider the semidirect product $\mathbb{C}^2 \rtimes \mathbb{R}$ with $\mathbb{R}$ acting by

$$\alpha_t(z, w) = (e^{2\pi i t} z, e^{2\pi i \theta t} w) \tag{63}$$

for fixed θ. If θ is irrational, then the orbits of this action are not closed.

If G is discrete, then $C^*(G)$ is GCR.

Naimark conjectured that if A is a C*-algebra with only one irreducible representation, then $A \cong B_0(H)$. This is true if A is separable. In the inseparable case, the answer is independent of ZF; the diamond principle can be used to construct a counterexample.

Given an ideal I, define $\operatorname{hull}(I) = \{J \in \hat{A} : J \supseteq I\}$, and givne a set $S \subseteq \hat{A}$, define $\ker(S) = \bigcap\{J \in S\}$.

Proposition 2.49. *(Spectral synthesis) For any ideal I, we have $I = \ker(\operatorname{hull}(I))$.*

Proof. Consider $B = A/I$, a C*-algebra. It has enough irreducible representations so that their direct sum is faithful. Each of these irreducible representations is an irreducible representation of A, and the intersection of their kernels, which are all primitive, is I. □

Definition For any ring R, not necessarily commutative, a *prime* ideal is a proper ideal P such that if J_1, J_2 are ideals with $J_1 J_2 \subseteq P$, then either $J_1 \subseteq P$ or $J_2 \subseteq P$.

Proposition 2.50. *Let A be a C*-algebra. Then any primitive ideal is prime.*

Proof. Let I be a primitive ideal. Then there is an irreducible representation (π, H) with kernel I. Let J_1, J_2 be ideals such that $J_1 J_2 \subseteq I$. Since I is closed, we can assume that J_1, J_2 are closed. If $J_1 \subseteq I$ we're done, so suppose otherwise. Then $\pi|_{J_1}$ is irreducible and nondegenerate, so $H = \overline{\pi(J_1)H}$. Then $\pi(J_2)H = \pi(J_2)\overline{\pi(J_1)H} = \overline{\pi(J_1 J_2)H} = 0$, from which it follows that $J_2 \subseteq I$. □

Is every closed prime ieal of a C*-algebra primitive? This is true in the separable case. In 2001 a non-separable counterexample was constructed.

Let R be a ring and Spec R its set of prime ideals. For any ideal I in R set $\operatorname{hull}(I) = \{J \in \operatorname{Spec} R : I \subseteq J\}$ and for any $S \subseteq \operatorname{Prime}(R)$ set $\ker(S) = \bigcap\{J \in S\}$. The hull-kernel or Jacobson topology on Spec R is defined using the closure operator $\bar{S} = \operatorname{hull}(\ker(S))$. When R is commutative this reproduces the Zariski topology.

Let A be a C^* algebra, and $\hat{A}$ its primitive ideal space with the hull-kernel topology. It is a T_0 topology, equivalently if $I, J \in \hat{A}$ and if $\overline{\{I\}} = \overline{\{J\}}$ then $I = J$, as $I \subseteq J$ and $J \subseteq I$. Can show that $\hat{A}$ is locally compact, i.e. for each point of $\hat{A}$ there is a compact neighborhood of the point. If A is separable then Baire category theorems work for $\hat{A}$.

3 Generators and relations

Given a set S whose elements are viewed as generators, for each $a \in S$ we want another symbol a^*. Consider the free algebra $F(S)$ over $\mathbb{C}$ for the generators S and S^* (multiplication is concatenation, with cancellation). Its elements are all the non-commutative polynomials in the a's and a^*'s. Define a $*$ on the free algebra in the evident way. Relations are *n.c.* polynomials in the a's and a^*'s. Let R be a set of relations, let $I(S, R) =$ the 2-sided $*$-ideal of $F(S)$ generated by R. Set $A(S, R) = F(S)/I(S, R)$: is a $*$-algebra. We can look at $*$-representations of $A(S, R)$ into $B(H)$, H Hilbert. For $a \in A(S, R)$ set $||a||_{C^*} = \sup\{||\pi(a)|| : \pi$ is a representation of $A(S, R)\}$. Can be $+\infty$ - if it does then C^* algebra does not exist. Eg $S = \{x\}$, $R = \emptyset$. We may have to mod out by things with norm zero, i.e. things in the kernel of every representation. Then we complete.

Themes:

1. Relations must force $|| \cdot || < \infty$. Suffices to show $||a||_{C^*} < \infty$ for each $a \in S$.
2. Are there many reps (relations like $x^*x = 0$ make the only rep trivial)
3. There may be a natural representations of $A(S, R)$. Is the operator norm for it $= || \cdot ||_{C^*}$

Example $S = \{u, 1\}$, $uu^* = 1 = u^*u$, $1u = u = u1$, $1^* = 1$.

Under any $*$-representation, the image of u is a unitary operator. This is $C(T)$. u corresponds to the function $f(z) = z$ on the circle.

Example $S = \{s, 1\}$, $s^*s = 1$, $1s = s1 = s$.

This gives $l^2(\mathbb{N})$. s seems to be a unilateral shift.

Example Let G be a group. Set $S = G, R = \{$ all relations for G, and $x^* = x^{-1}$ for all $x \in G\}$.

We always have the trivial representation and the left regular representation $\lambda : l^2(G)$. $(\lambda_x \xi)(y) = \xi(x^{-1}y)$ for $\xi \in l^2(G)$. $A(G, R) = \{\sum_{\text{finite}} f(x)x, f \in C_c(G)\}$. Let $||f||_r = ||\lambda_{\sum_{f(x)x}}||$. Is it true that $||f||_r = ||f||_{C^*}$? No. True exactly if G is "amenable."

Example Abelian groups, solvable groups, nilpotent groups, finite groups are amenable. $SL_n(\mathbb{Z})$ is not amenable.

G (discrete) group. $f \in C_c(G)$ are functions of finite support. Given (π, H) a unitary representation of G, set $(\pi_f \xi) = \sum_{x \in G} f(x)\pi_x \xi, \xi \in H$.

$$||\pi_f \xi|| \leq \sum |f(x)| ||\pi_x \xi|| = ||f||_1 ||\xi||$$

(used unitary to conclude $||\pi_x \xi|| = ||\xi||$.) So

$$||\pi_f|| \leq ||f||_1.$$

Extend def to $l^1(G), \pi_f \xi$ to $f \in l^1(G)$. Given $f, g \in C_c(G)$,

$$\begin{aligned} \pi_f(\pi_g \xi) &= \sum_x f(x)\pi_x(\sum_g g(y)\pi_y \xi) \\ &= \sum_x f(x)(\sum g(y)\pi_{xy}\xi) \\ &= \sum_y (\sum_x f(x)g(x^{-1}y))\pi_y \xi \end{aligned}$$

So define

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y).$$

We have

$$||f * g||_1 \leq ||f||_1 ||g||_1$$

and

$$\pi_f \pi_g = \pi_{f*g}$$

. We want our representation to be a $*$ representation

$$(\pi_f)^* = \sum \overline{f(x)}\pi_x^* = \sum \overline{f(x^{-1})}\pi_x.$$

So define $f^*(x) = \overline{f(x^{-1})}$. Then $(\pi_f)^* = \pi_{f^*}$. $G \subset l^1(G)$ via $x \mapsto \delta_x$ and we have $\pi_{\delta_x} = \pi_x$.

Proposition 3.1. *There is a bijection between the unitary reps of G and the non-degenerate $*$-representations of $l^1(G)$ as defined above.*

Proof. We've done one direction. Given (π, H) a non-degenerate $*$-representation of $l^1(G)$, set $\pi_x = \pi_{\delta_x}$. $\square$

For $f \in l^1(G)$, set

$$||f||_{C^*} = \sup\{||\pi_f|| : (\pi, H) \text{ a unitary rep of } G\}$$

We have $||f||_{C^*} \leq ||f||_1$. Question do we always have a unitary rep of G? There is a trivial one which isn't much help. Always have "left-regular" representation of G. $(l^2(G), \lambda), (\lambda_x \xi)(y) = \xi(x^{-1}y)$. For $f \in l^1(G)$, $\lambda_f(\delta_e) = f \in l^2(G)$. Thus if $\lambda_f = 0$ then $f = 0$, so λ is faithful. So can define

$$||f||_{C^*_r} = ||\lambda_f|| \leq ||f||_{C^*}.$$

Theorem 3.2. $||f||_{C^*_r} = ||f||_{C^*}$ *for all* $f \in l^1(G)$ *off* G *is amenable.*

Definition For any set X, a positive linear functional on $l^\infty(X)$ is called a mean. For G discrete, say that G is amenable if there is a mean on $\mu \in l^\infty(G)$, $||\mu|| = 1$ that is invariant under left translation.

All commutative groups are amenable. Prove that $\mathbb{Z}$ is amenable. $SL_n(\mathbb{Z})$ is not amenable for $n \geq 2$. Free group on n-letters is not amenable.

4 Tensor products

Let A and B be unital C^* algebras. Consider $A \otimes B$. Generators $A \cup B$. Relations: all relations in A, B $ab = ba$ for all $a \in A, b \in B$. $1_A = 1_B$. ab denotes $a \otimes b$. $(a \otimes b)^* = a^* \otimes b^*$. $A \hookrightarrow A \otimes B$ via $a \mapsto a1_B$.

For any $*$-rep (π, H) of $A \otimes B$, $||\pi(a \otimes 1_B)|| \leq ||a||$, $||\pi(1_A \otimes b)|| \leq ||b||$. $||a \otimes b|| \leq ||a|| ||b||$ " cross-nom ".

Is this just the zero algebra? Let (π, H) be a $*$-rep of A, ρ, K a $*$-rep of B. Form $(\pi \otimes \rho$ on $H \otimes K)$. For $H \otimes K$ as a vector space.

$$< \xi \otimes \eta, \xi' \otimes \eta' > := < \xi, \xi' >_H < \eta, \eta' >_K$$

and extend by linearity. Then complete to get a Hilbert space. Given $S \in B(H)$, set $(S \otimes I_K)(\xi \otimes \eta) := S\xi \otimes \eta$. $||S \otimes I_K|| = ||S||$. For $T \in B(K)$, set $S \otimes T = (S \otimes I_K)(I_H \otimes T)$, then $||S \otimes T|| = ||S||||T||$.

Let A, B be unital C*-algebras. If (π, H) and (ρ, K) are representations of A and B, then we can define a tensor product $A \otimes B$ of *-algebras which naturally acts on the Hilbert space tensor product $H \otimes K$ by

$$(\pi \otimes \rho)(a \otimes b) = \pi(a) \otimes \rho(b) \tag{64}$$

and then extending by linearity. On the algebraic tensor product $A \otimes B$ we can define a norm

$$\|t\| = \sup\{\|(\pi \otimes \rho)(t)\| : (H, \pi) \text{ a rep of } A, (\rho, K) \text{ a rep of } B\}. \tag{65}$$

and if $t = \sum a_j \otimes b_j$ then this is at most $\sum \|a_j\|\|b_j\|$. Completing $A \otimes B$ with respect to this norm gives a C*-algebra. We will denote this by $A \otimes_{\min} B$ (it turns out to be the minimal norm that can be placed on the tensor product). The maximal norm, which gives an algebra $A \otimes_{\max} B$, has norm

$$\|t\| = \sup\{\|(\pi \otimes \rho)(t)\| : \pi, \rho \text{ reps on } H, \pi(a)\rho(b) = \pi(b)\rho(a)\}. \tag{66}$$

An example where the two differ is as follows. Let G be a discrete group, let $H = \ell^2(G)$, and consider the left regular representation λ and the right regular representation ρ of $C_r^*(G)$. If $\sigma = \lambda \otimes \rho$, then $\|\sigma(t)\|$ is strictly bigger than $\|t\|_{\min}$ in general (Takesaki showed that this was true for $G = F_2$). We also know that $C_r^*(F_2)$ and $C_r^*(F_2) \otimes_{\min} C_r^*(F_2)$ are simple. On the other hand, $C^*(F_2) \otimes C^*(F_2)$ contains a copy of $B_0(\ell^2(F_2))$.

Definition A C*-algebra A is *nuclear* if, for all C*-algebras B, we have $A \otimes_{\min} B = A \otimes_{\max} B$.

Example All GCR algebras are nuclear. Any filtered colimit of nuclear algebras is nuclear.

Theorem 4.1. *Let G be a discrete group. $C_r^*(G)$ is nuclear iff G is amenable (iff $C_r^*(G) = C^*(G)$).*

This is false if we do not assume that G is discrete.

Theorem 4.2. *$0 \to I \to A \to A/I \to 0$ is a short exact sequence (where I is a two-sided ideal), then for any B, the tensor product*

$$0 \to I \otimes_{max} B \to A \otimes_{max} B \to (A/I) \otimes_{max} B \to 0 \tag{67}$$

is also a short exact sequence.

(This fails for the minimal tensor product.) If this holds for the minimal tensor product for fixed B and all A, I then we say that B is *exact*. (Hence nuclear implies exact.)

We can also form free products $A * B$.

5 Group actions

Let G be a discrete group. Suppose we are given an action $\alpha : G \to \mathrm{Aut}(A)$ where A is a C*-algebra. We may think of this as a generalized dynamical system; in the special case that $A = C_0(X)$ we take actions $\alpha : G \to \mathrm{Homeo}(X)$. In the noncommutative case, we call such things C*-dynamical systems.

A covariant representation of a C*-dynamical system (A, G, α) is a triple (H, π, U) where π is a representation of A on H and U is a unitary representation of G such that

$$\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}. \tag{68}$$

There is a universal C*-algebra here whose representations correspond to covariant representations of a given C*-dynamical system. If $f \in C_c(G, A)$ is a function on G with finite support and values in A and (H, π, U) is a covariant representation, we define

$$\sigma_f v = \sum \pi(f(x)) U_x v \tag{69}$$

where $v \in H$. This gives

$$\sigma_f \sigma_g v = \sum_x \pi(f(x)) U_x \sum \pi(g(y)) U_y v = \sum_{x,y} \pi(f(x)) \pi(\alpha_x(g(y)) U_{xy} v. \tag{70}$$

We can reindex this sum, which allows us to define a product on $C_c(G, A)$ by

$$(f * g)(y) = \sum_x f(x)\alpha_x(g(x^{-1}(y))). \tag{71}$$

Note that if the action is trivial then this is just ordinary convolution. We define a norm $\|f\|_1 = \sum_x \|f(x)\|$; with respect to this norm, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Alternately, we can think about the *-algebra freely generated by A and the elements of G subject to the relations $x^* = x^{-1}$ and $\alpha_x(a) = xax^{-1}$. The corresponding universal C*-algebra is denoted $C^*(A, G)$ or $A \rtimes_\alpha G$ and is called the *covariance C*-algebra* or *crossed product* for the C*-dynamical system.

Are there any nontrivial covariance representations? Let (ρ, H_0) be a representation of A and let $H = \ell^2(G) \otimes H_0$ (equivalently, square-integrable functions on G with values in H_0). Let $(U_x v)(y) = v(x^{-1}y)$ where $v \in \ell^2(G, H_0)$ and let

$$(\pi(a)v)(x) = \rho(\alpha_x^{-1}(a))v(x). \tag{72}$$

We can verify that this is a covariant representation. If ρ is a faithful representation of A, then H is a faithful representation of $C_c(G, A)$. So let

$$\|f\|_{C_r^*} = \sup\{\|\pi(f)\|\} \tag{73}$$

where π varies over all representations constructed above. We say that α is *amenable* if $C_r^*(A, G, \alpha) = C^*(A, G, \alpha)$. If G is amenable, then α is amenable, but not conversely.

If H is a subgroup of G and we are given a covariant representation of $(A, H, \alpha|_H)$, then by replacing $\ell^2(G)$ with the action of G on $\ell^2(G/H)$, we can obtain a covariant representation of (A, G, α) called the *induced representation*.

If A has an identity, then given a nondegenerate representation of $C^*(A, G, \alpha)$, we can restrict it to A and G to get a covariant representation of (A, G, α). If A has no identity, we pass to the multiplier algebra of $C^*(A, G, \alpha)$. In both cases, we get a natural bijection.

If G is a topological group, we want to consider unitary representations of G which are strongly continuous (continuous in the strong operator topology), e.g. for every $v \in H$ the function $x \mapsto U_x v$ is continuous. Continuity is too strong a condition; for example, the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ fails to be norm-continuous. We also require strong continuity for actions of G on C*-algebras and strong continuity for covariant representations of a C*-dynamical system.

The theory is nicest when G is locally compact (Hausdorff). Such groups have a (left) Haar measure (nonzero positive Radon measure (positive linear functional on $C_c(G)$) which is invariant under left translation), so one can talk about the corresponding L^p spaces. Taking inverses gives us a right Haar measure which is invariant under right translation; they need not be the same in general (e.g. for the $ax + b$ group). The Radon-Nikodym derivative is a continuous homomorphism $\Delta : G \to \mathbb{R}_{>0}$, the *modular function*, such that

$$\int_G f(x^{-1})\, dx = \int_G f(x)\Delta(x)\, dx. \tag{74}$$

In particular, if G is compact, $\Delta = 1$ identically (G is *unimodular*) and the left and right Haar measures agree. Semisimple and nilpotent Lie groups are also unimodular, but some solvable groups are not. Discrete groups and commutative groups are also unimodular.

The action of G on $L^p(G)$ is strongly continuous for finite p. To see this, the action of G on $C_c(G)$ is strongly continuous, and this is dense in $L^p(G)$ for finite p. We can define convolution on $L^1(G)$ by

$$(f * g)(x) = \int f(y)g(y^{-1}x)\, dy \tag{75}$$

and this gives a Banach algebra.

If (H, U) is a strongly continuous unitary representation of G, we can define

$$U_f v = \int f(x)U_x v\, dx \tag{76}$$

where $f \in L^1(G)$, and this gives a non-degenerate *-representation of $L^1(G)$.

If (A, G, α) is a C*-dynamical system and $f, g \in C_c(G, A)$ we can again define

$$(f * g)(x) = \int f(y)\alpha_y(g(y^{-1}x))\, dy \tag{77}$$

and again if (H, π, U) is a covariant representation we can define

$$\sigma_f v = \int \pi(f(x))U_x v\, dx \tag{78}$$

which gives a continuous action of $C_c(G, A)$ on H. The details are the same as in the discrete case except that

$$
\begin{aligned}
(U_f)^* &= \left(\int f(x)U_x\,dx\right)^* & (79)\\
&= \int \bar{f}(x)U_x^*\,dx & (80)\\
&= \int \bar{f}(x)U_{x^{-1}}\,dx & (81)\\
&= \int \bar{f}(x^{-1})U_x\Delta(x^{-1})\,dx & (82)
\end{aligned}
$$

so we need to define

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \quad (83)$$

For $f \in C_c(G, A)$ define as before

$$\|f\|_{C^*} = \sup\{\|\sigma_f\|\} \quad (84)$$

where σ_f comes from a covariant representation of (A, G, α). Completing with respect to this norm gives a C*-algebra $C^*(A, G, \alpha)$, often written $A \rtimes_\alpha G$ as before. We can also restrict our attention to representations induced from representations of A and then we get the reduced algebra $C_r^*(A, G, \alpha)$. If α is trivial, then $C^*(A, G, \alpha) = A \otimes_{\max} C^*(G)$ and $C_r^*(A, G, \alpha) = A \otimes_{\min} C_r^*(G)$.

Given a representation of $L^1(A, G, \alpha)$, do we get a covariant representation of (A, G, α)? For G discrete and A unital we consider the inclusions

$$G \ni g \mapsto \delta_g \in \ell^1(A, G, \alpha) \quad (85)$$

and

$$A \ni a \mapsto \delta_1(a) \in \ell^1(A, G, \alpha) \quad (86)$$

and this gives the correspondence. In general, let e_λ be a 2-sided approximate identity of norm 1 in A and let f_μ be an approximate δ-function for $L^1(G)$ in $C_c(G)$. Set

$$g_{\delta,\mu}(x) = f_\mu(x)e_\lambda \in C_c(G, A). \quad (87)$$

Check that $\delta_x * f = \lambda_x f$ and $f \circ \delta_x = f\rho_x$ (need to put Δ in here somewhere). There is an inclusion

$$L^1(A, G, \alpha) \to L^1(\tilde{A}, G, \alpha) \oplus \ell^1(\tilde{A}, G_d, \alpha) \tag{88}$$

where $\tilde{A}$ is the unitalization and G_d is G with the discrete topology. Any nondegenerate representation of the LHS extends to a nondegenerate representation of the RHS on the same Hilbert space, which gives the following.

Theorem 5.1. *There is a bijection between the nondegenerate representations of $L^1(A, G, \alpha)$ (and also $C^*(A, G, \alpha)$) and the covariant representations of (H, π, U) of (A, G, α) given by forming the integrated form.*

Let G be a group acting on C*-algebras A and B. We want to consider equivariant *-homomorphisms $\Phi : A \to B$. Given such a map, we can define $\hat{\Phi} : L^1(A, G, \alpha) \to L^1(B, G, \beta)$ by

$$(\hat{\Phi}(f))(x) = \Phi(f(x)) \tag{89}$$

for $f \in L^1(A, G, \alpha), x \in G$. Then $\hat{\Phi}$ is a *-homomorphism.

Note that $L^1(A, G, \alpha)$ embds into $C^*(A, G, \alpha)$. Given a nondegenerate representation of $C^*(B, G, \beta)$, we get a representation of $L^1(A, G, \alpha)$ by pulling back, but it is not necessarily nondegenerate.

Let (A, G, α) be a C*-dynamical system and J a 2-sided closed ideal. If J is α-invariant, then (J, G, α) and $(A/J, G, \alpha)$ are C*-dynamical systems.

Theorem 5.2. *The sequence*

$$0 \to J \rtimes_\alpha G \xrightarrow{i_*} A \rtimes_\alpha G \xrightarrow{p_*} A/J \rtimes_\alpha G \to 0 \tag{90}$$

is exact.

Proof. p_* being a surjection is clear. The composition of the maps being zero is clear. To show injectivity of i_*, let σ be a faithful nondegenerate representation of $J \rtimes_\alpha G$ and let (H, π, U) be the corresponding covariant representation.

Lemma 5.3. *If σ is the integrated form of π, then σ is nondegenerate iff π is nondegenerate.*

Proof. Let $N = \{v \in H : \pi(b)v = 0 \forall b \in V\}$. Then N is U-invariant, so $\sigma(J)v = 0$ for $v \in N$. Then π extends to a representation $\tilde{\pi}$ of A on H with $\tilde{\pi}(a)(\pi(d)v) = \pi(ad)v$, and $(H, \tilde{\pi}, U)$ is a covariant representation of (A, G, α). Let $\tilde{\sigma}$ be the integrated form of $(H, \tilde{\pi}, U)$. Then we claim that $\tilde{\sigma} \circ i_* = \sigma$ where σ is faithful. □

It follows that i_* is injective.

To show exactness, view $J \rtimes_\alpha G$ as contained in $A \rtimes_\alpha G$. We know that $p_*(J \rtimes_\alpha G) = 0$, so $\ker(p_*) \subseteq J \rtimes_\alpha G$, and we want equality.

Let σ be a faithful representation of $A \rtimes_\alpha G / J \rtimes_\alpha G$. View σ as a representation of $A \rtimes_\alpha G$ with kernel $J \rtimes_\alpha G$. Let (H, π, U) be the corresponding covariant representation for σ (of (A, G, α)). For any $d \in J$ and any $h \in C_c(G, \mathbb{C})$, let $J(x) = h(x)d \in J \rtimes_\alpha G$. Then

$$0 = \sigma_J v = \int \pi(J(x)) U_x v \, dx = \int \pi(d) h(x) U_x v \, dx = \pi(d) U_h v \tag{91}$$

for all h, hence $\pi(d) = 0$, hence $\pi(J) = 0$. View π as a representation of A/J, call it $\hat{\pi}$. Then $(H, \hat{\pi}, U)$ is a covariant representation of $(A/J, G, \alpha)$. Let $\hat{\sigma}$ be its integrated form, a representation of $A/J \rtimes_\alpha G$. We claim without proof that $\hat{\sigma} p_* = \sigma$, thus $\ker(p_*) \subseteq J \rtimes_\alpha G$. □

5.1 Transformation groups

Now let A be commutative, so $A \cong C_0(M)$. Then G acts continuously on M (a transformation group). In this case $A \rtimes_\alpha G$ is called a transformation group algebra.

Theorem 5.4. *Let σ be an irreducible representation of $C_0(M) \rtimes_\alpha G$. Let (H, π, U) be the corresponding covariant representation. Let*

$$J = ker(\pi) = \{F \in C_0(M) : F(m) = 0 \forall m \in Z_J \subset M\} \tag{92}$$

where $Z_J = hull(J) = \{m \in M : f(m) = 0 \forall f \in J\}$ (which is α-invariant). Let M be second-countable. Then Z_J is the closure of the orbit of some point of M.

Proof. Fix $v \in H$ with $\|v\| = 1$. Define a finite Radon measure μ on M by

$$\mu(f) = \langle \pi(f)v, v \rangle \tag{93}$$

for $f \in A$. We have $\mu(f) = 0$ for $f \in J$, so the support of μ is contained in Z_J. Let B be a countable basis for the topology of M and let B_n be an enumeration of the elements of B. For $n \in \mathbb{N}$, let

$$O_n = \alpha_G(B_n) = \bigcup_{x \in G} \alpha_x(B_n) \tag{94}$$

which is α-invariant and open. Let $J_n = C_0(O_n) \subseteq C_0(M)$. If $B_n \cap Z_J \neq \emptyset$, then $O_n \cap Z_J \neq \emptyset$, so $J_n \nsubseteq J$, whence $\pi(J_n) \neq \{0\}$. Now $\overline{\pi(J_n)H}$ is invariant under $U_x, x \in G$ and $\pi(a), a \in A$, so it is invariant under the integrated form of (H, π, U). Since H is irreducible, it must be zero or H. Hence it is H if $B_n \cap Z_J \neq \emptyset$, i.e. if $\pi|_{J_n}$ is nondegenerate.

Let $S \subseteq \mathbb{N}$ be the set of $n \in \mathbb{N}$ such that $B_n \cap Z_J \neq \emptyset$. Given $n \in S$, take an approximate identity e_λ for J_n, whence

$$\lim_\lambda \mu(e_\lambda) = \lim_\lambda \langle e_\lambda v, v \rangle = 1 \tag{95}$$

so $\mu(O_n) = 1$. Thus $\mu(\bigcap O_n) = 1$, since μ is a probability measure and S is countable. Hence $Z = Z_J \cap \bigcap O_n \neq \emptyset$.

We claim that if $m \in Z$ then the orbit of m is dense in Z_J. To see this, let $m_0 \in Z_J$ and let U be an open neighborhood of m_0. Then $m_0 \in B_n \subseteq U$ for some $n \in S$. Since $m \in O_n = \alpha_G(B_n)$, there exists $x \in G$ such that $\alpha_x(m) \in B_n \subseteq U$.

Hence Z_J is the closure of the orbit of m for any $m \in Z$. $\square$

Example Let $G = \mathbb{Z}, M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\theta \in \mathbb{R} \backslash \mathbb{Q}$ and let α be the homeomorphism of $\mathbb{T}$ given by translation by θ. Then the orbit of any point of $\mathbb{T}$ is a countable dense subset of $\mathbb{T}$. We may define

$$A_\theta = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} \tag{96}$$

(an irrational rotation C*-algebra), which is an example of a noncommutative torus. A_θ is generated by U and V such that

$$VU = e^{2\pi i \theta} UV \tag{97}$$

(where $V = e^{2\pi i t}$ generates $C(\mathbb{T})$). Now $C(\mathbb{T})$ acts on $C_b(\mathbb{Z})$. This action, and translation by $\mathbb{Z}$, gives an irreducible representation of A_θ.

We can also stick in cocycles to get other irreducible representations.

Let G be a locally compact group, $A = C_0(G)$ and α be the action of G on A by left translation. We can form the C*-algebra $C^*(G, C_0(G), \alpha) = C_0(G) \rtimes_\alpha G$.

Theorem 5.5. *$C_0(G) \rtimes_\alpha G$ is naturally isomorphic to $B_0(L^2(G))$.*

Proof. $C_0(G) \rtimes_\alpha G$ has a natural covariant representation (π, U) on $L^2(G)$ (which we might call the Schrodinger map). Let σ be its integrated form, defined on $C_c(G, C_0(G)) \supseteq C_c(G \times G)$ by

$$(\sigma_F(\xi))(x) = (\int F(y) U_y \xi)(x) = \int F(y, x) \xi(y^{-1}x)\, dy. \tag{98}$$

Let $f, g \in C_c(G) \subset L^2(G)$ and $\xi \in C_c(G)$. Let $\langle f, g\rangle_0$ be the rank one operator given by

$$\langle f, g\rangle_0 \xi = f \langle g, \xi\rangle_{L^2(G)}. \tag{99}$$

Define

$$\langle f, g\rangle_E(y, x) = f(x)\bar{g}(y^{-1}x)\Delta(y^{-1}x) \in C_c(G \times G) \tag{100}$$

so that $\sigma_{\langle f,g\rangle_E} = \langle f, g\rangle_0$. Let E be the linear span of the functions $\langle f, g\rangle_E$ for $f, g \in C_c(G)$. Then E is stable under pointwise product and complex conjugation, and moreover it separates the points of $G \times G$. Hence E is dense in $C_c(G \times G)$ in the colimit topology, so E is dense in $L^1(G, C_0(G))$ and so in $C_0(G) \rtimes_\alpha G$.

If $f_1, ...f_n \in C_c(G)$ are orthonormal, then the $\langle f_j, f_k\rangle_E$ span a copy of $\mathcal{M}_n(\mathbb{C})$ (hence a C*-algebra with a unique C*-norm) inside $C_0(G) \rtimes_\alpha G$. Hence on this span the norm for $C_0(G) \rtimes_\alpha G$ agrees with the norm on $B(L^2(G))$ via σ. Hence σ is isometric on E. Hence σ is isometric on $C_0(G) \rtimes_\alpha G$ and maps into $B_0(L^2(G))$, and we saw that it's onto. □

More generally, we can consider $C_0(G/H) \rtimes_\alpha G$, which turns out to be Morita equivalent to $C^*(H)$.

Since the reduced cross product is a quotient of the cross product $C_0(G) \rtimes_\alpha G$, which is $B_0(L^2(G))$, and since $B_0(L^2(G))$ has no proper quotients, we conclude that α is amenable.

Given groups Q, N and $\alpha : Q \to \mathrm{Aut}(N)$ an action, we can form the semidirect product $G = N \rtimes_\alpha Q$, which is $N \times Q$ with the multiplication given by

$$(n, x)(m, y) = (n\alpha_x(m), xy). \tag{101}$$

We can do this for topological groups as well. These groups fit together in a split exact sequence

$$0 \to N \to G \to Q \to 0. \tag{102}$$

Let N, Q be locally compact. If (H, U) is a strongly continuous unitary representation of G, then it restricts to unitary representations $U|_N, U|_Q$ of N and Q with a covariance relationship. $U|_N$ has an integrated form σ^N giving a representation of $C^*(N)$ on H. For any $x \in Q$, α_x is an automorphism of N, so this gives an automorphism of $L^1(N)$. Furthermore, via α, Q acts on the set of unitary representations of N, so acts via a group of automorphisms of $C^*(N)$. This action is strongly continuous, so we can form the crossed product

$$C^*(N) \rtimes_\alpha Q. \tag{103}$$

We find that $(H, \sigma^N, U|_Q)$ is a covariant representation of $(C^*(N), Q, \alpha)$, hence gives a representation of the crossed product. The converse also holds.

Proposition 5.6. *There is a natural isomorphism $C^*(N \rtimes_\alpha Q) \cong C^*(N) \rtimes_\alpha Q$.*

If N is commutative, then $C^*(N)$ is commutative, so $C^*(N) \cong C_0(\hat{N})$ where $\hat{N}$ is the Pontrjagin dual group of all continuous homomorphisms $N \to \mathbb{T}$. Then Q acts on $\hat{N}$, and

$$C^*(N \rtimes_\alpha Q) \cong C_0(\hat{N}) \rtimes_\alpha Q. \tag{104}$$

Wigner in 1936 was the first to explore these issues. Consider $\mathbb{R}^4$ equipped with the bilinear form

$$B(v, w) = v_0w_0 - v_1w_1 - v_2w_2 - v_3w_3. \tag{105}$$

The Lorentz group L is the group of linear transformations on $\mathbb{R}^4$ preserving B. Let α be its action on $\mathbb{R}^4$. The Poincaré group is the semidirect product $P = \mathbb{R}^4 \rtimes_\alpha L$, and we want to consider its (physically interesting) unitary representations. Since $\hat{\mathbb{R}^4} \cong \mathbb{R}^4$, we are looking at $C_0(\mathbb{R}^4) \rtimes_\alpha L$, and we need irreducible representations of L.

For $v \in \hat{\mathbb{R}^4} \neq 0$, consider the stabilizer P_v. The orbit of v looks like P/P_v, so we want irreducible representations of $C_0(P/P_v) \rtimes_\alpha L$. These representations correspond to representations of the little group L_v. For massive particles, $L_v = \mathrm{SU}(2)$.

Given any group G and subgroup H we know that $C_0(G/H) \rtimes_\alpha G \cong C^*(H)$. Let's be explicit about this. Given a representation $(\mathcal{H}, U)$ of H, we get a representation of $C_0(G/H) \rtimes_\alpha G$ by constructing the induced representation

$$\mathcal{K} = \{\xi : G \to H : \xi(xs) = U_s(\xi(x))\} \tag{106}$$

where $s \in H, x \in G$. Note that $x \mapsto \|\xi(x)\|^2$ is H-invariant so can be identified with a function on G/H; we require that this function is integrable.

6 Quantum mechanics

To model a quantum-mechanical system, observables are modeled by self-adjoint operators on a Hilbert space. For a given observable, the collection of possible numbers obtained by observation should be the spectrum of the operator. States are modeled by expected values. Pure states are modeled by rank-1 projections.

If P is a rank-1 projection and T is a self-adjoint operator, then the expected value when we observe the state P for the observable T is

$$\mathrm{tr}(T^*P) = \mathrm{tr}(TP) = \langle Tv, v \rangle \tag{107}$$

(where $P = vv^*, \|v\| = 1$).

Mixed states are modeled by positive operators B of trace 1, and the corresponding expected value is $\mathrm{tr}(TB)$.

Heisenberg showed that if Q is the position operator in a certain direction for a particle and P its momentum, then

$$[P, Q] = -i\hbar \tag{108}$$

where $\hbar$ is Planck's constant. This implies that the Hilbert space must be infinite-dimensional, since if P, Q are finite-rank operators then $\mathrm{tr}([P, Q]) = 0$.

For n particles we need $3N$ such pairs of operators P_i, Q_i where $i \in \{1, ...3n\}$ for each space direction.

Weyl set $U_s = e^{is\mathrm{Re}(P)}$, which gives a 1-parameter strongly continuous family of unitary operators. Similarly, let $V_t = e^{it\mathrm{Re}(Q)}$. What happens to the Heisenberg commutation relation? Heuristically, let

$$f(s) = U_s Q U_s^* = e^{isP} Q e^{-isP}. \tag{109}$$

Then

$$f'(s) = ie^{isP}(PQ - QP)e^{-isP} = \hbar. \tag{110}$$

Since $f(0) = Q$, we have $f(s) = Q + s\hbar$. Then

$$U_s V_t U_s^* = e^{it(U_s Q U_s^*)} = e^{is\hbar t} V_t \tag{111}$$

and we get the Weyl commutation relations

$$U_s V_t = e^{ihst} V_t U_s. \tag{112}$$

In general, let G be a locally compact (Hausdorff) abelian group. A *representation of the Heisenberg commutation relations* is a triple (H, U, V) such that U is a (strongly continuous) unitary representation of G on H, V is a unitary representation of $\hat{G}$ on H, and

$$U_s V_t = \langle s, t\rangle V_t U_s. \tag{113}$$

For example, we can take $H = L^2(G)$ with U_s translation by s and V_t pointwise multiplication by $s \mapsto \langle s, t\rangle$.

V has an integrated form as a representation of $L^1(\hat{G})$, hence of $C^*(\hat{G}) \cong C_0(G)$ (by Pontrjagin duality). Let $\varphi \in L^1(\hat{G})$ and $f = \hat{\varphi}$, so that

$$\pi(f) = \int \varphi(t) V_t \, dt. \tag{114}$$

Then

$$U_x \pi(f) U_x^{-1} = \int \varphi(t) U_x V_t U_x^{-1} \, dt = \pi(\alpha_x(f)) \tag{115}$$

(where α_x is the action by translation). It follows that (H, π, U) is a covariant representation of $(C_0(G), G,]alpha)$, and hence gives a representation of $C_0(G) \rtimes_\alpha G \cong$

$B_0(L^2(G))$.

Theorem 6.1. *Suppose that G is a locally compact (Hausdorff) abelian group. Then up to unitary equivalence there is only one irreducible representation of the Weyl form of the canonical commutation relations, namely the Schrödinger representation, and every representation is a direct sum of copies of this.*

For the ordinary Heisenberg commutation relations we have the same story.

6.1 Projective representations

Let G be a locally compact group and let $H = G \times \hat{G}$. Let $W_{(x,t)} = U_x V_t$. Then

$$W_{(x,s)} W_{(y,t)} = \overline{\langle y, s\rangle} W_{(x,s)+(y,t)}. \tag{116}$$

In terms of quantum mechanics, pure states can be identified with the projective space of the underlying Hilbert space, so symmetries sould be governed by automorphisms of this projective space. Wigner showed that every automorphism of projective Hilbert space is given by either a unitary operator or an anti-unitary operator.

This motivates the following definition. We say that $\omega : G \to \mathrm{U}(H)$ is a *projective representation* of G if there exists $c : G \times G \to \mathbb{T}$ such that

$$\omega_x \omega_y = c(x, y)\omega_{xy} \tag{117}$$

for $x, y \in G$. Associativity implies that

$$c(xy, z)c(x, y) = c(x, yz)c(y, z) \tag{118}$$

hence that c is a 2-cocycle with values in $\mathbb{T}$. We can normalize c so that $c(e, x) = c(x, e) = 1$.

Given $h : G \to \mathbb{T}$, define $(\delta h)(x, y) = h(xy)h(x^{-1})h(y^{-1})$. We say that two cocycles c, c' are cohomologous if $c = (\delta h)c'$. Projective representations of G associated to c, c' can then be canonically identified.

Projective representations of G associated to a fixed cocycle c correspond to representations of $L^1(G, c)$ (which is invariant under replacing c wiht a cohomologous cocycle c'), which is $L^1(G)$ with a twisted convolution

$$(f *_c g)(x) = \int f(y)g(y^{-1}x)c(y, y^{-1}x)\, dy. \tag{119}$$

This also works on $L^2(G)$, so we have twosted C*-algebras $C^*(G,c)$ and $C^*_r(G,c)$.

Continuing from above for $H = G \times \hat{G}$, we have $C^*(H,c) \cong B_0(L^2(G))$.

Given a Hilbert space H, the automorphism group $\mathrm{Aut}(B_0(H))$ consists of conjugation by a unitary or antiunitary operator. So actions on $B_0(H)$ are projective representations.

Given G and c, we can construct an extension group E_c fitting into a short exact sequence

$$0 \to \mathbb{T} \to E_c \to G \to 0 \tag{120}$$

such that the projective c-representations of G correspond to the ordinary representations of E_c which restrict to the trivial representation on $\mathbb{T}$.

Let $G = \mathbb{Z}^d$. Choose a $d \times d$ real matrix θ and define a bicharacter

$$c_\theta(m,n) = e^{2\pi i m \cdot \theta n} \tag{121}$$

where $\cdot$ denotes the dot product and $m, n \in G$. Every 2-cocycle with values in $\mathbb{T}$ is cohomologous to some c_θ.

Form $\ell^1(G, c_\theta)$. The delta functions $\delta_n, n \in G$ satisfy

$$\delta_m * \delta_n = c_\theta(m,n)\delta_{m+n} \tag{122}$$

so in particular $\delta_m * \delta_{-m} = c_\theta(m,-m)\delta_0$, hence

$$(\delta_m)^{-1} = c_\theta(m,m)\delta_{-m}. \tag{123}$$

We want to put a *-structure on this algebra such that each δ_n is unitary, so define $(\delta_m)^* = c_\theta(m,m)\delta_{-m}$. Then

$$\delta_n * \delta_m = c_\theta(n,m)\delta_{m+n} = \overline{c_\theta(m,n)} c_\theta(n,m) \delta_m * \delta_n \tag{124}$$

where the coefficient on the RHS simplifies to $e^{2\pi i n \cdot (\theta - \theta^t) m}$.

If $\theta = 0$, then $C^*(G, c_\theta) = C^*(G) = C(\mathbb{T}^d)$. Hence we will think of the C*-algebras $A_\theta = C^*(\mathbb{Z}^d, c_\theta)$ as noncommutative tori.

If $G = \mathbb{Z}^d$, then $\hat{G} = \mathbb{T}^d$ acts as a group of automorphisms of A_θ as follows: given $t \in \mathbb{T}^d$,

$$\alpha_t(\delta_m) = \langle m, t\rangle \delta_m. \tag{125}$$

This action is strongly continuous. The A_θ form the simplest interesting examples of noncommutative differentiable manifolds.

7 Strongly continuous group actions

Let U be a Banach space and α a strongly continuous action of $\mathbb{T}^d$ on U. For any $u \in U, n \in \mathbb{Z}^d$, the Fourier coefficient of u for n is

$$u_n = \int_G \overline{\langle t, n\rangle} \alpha_t(u)\, dt \in U. \tag{126}$$

We have $\alpha_s(u_n) = \langle s, n\rangle u_n$. The set U_n of u_ns for a fixed n is a closed subspace of U called the n-isotypic component of U for α.

Let G be a compact abelian group, e.g. T^d. Let $\hat{G}$ denote its Pontrjagin dual, e.g. $\mathbb{Z}^d$. We will denote the pairing between the two groups by $\langle t, m\rangle$. Let A be a C*-algebra and α an action of G on A which is strongly continuous. For $a \in A$, let

$$a_n = \int_G \overline{\langle t, n\rangle} \alpha_t(a)\, dt \tag{127}$$

where $\int_G$ is normalized Haar measure. Let $e_n(t) = \overline{\langle t, n\rangle} \in L^1(G)$. Then

$$(e_m * e_n)(t) = \langle t, n\rangle \int_G \overline{\langle s, m-n\rangle}\, ds. \tag{128}$$

This is equal to zero whenever $m - n \neq 0$, and is equal to e_n if $m = n$. That is,

$$e_m * e_n = \delta_{mn} e_n. \tag{129}$$

So e_n is a self-adjoint projection in $L^1(G)$.

Let $A_n = \{a : \alpha_t(a) = \langle t, n\rangle a\} = \alpha_{e_n}(a)$. Then for $m \neq n$ we have $A_n \cap A_m = \{0\}$.

Proposition 7.1. *The algebraic direct sum $\bigoplus_{n\in\hat{G}} A_n$ is dense in A.*

Proof. The linear span of the e_n is a *-subalgebra of $C(G)$ with $e_0 = 1$. It separates points, so is dense in $C(G)$. Hence we can find an approximate identity i_λ in this dense span. □

One way of stating this result is that A is graded over $\hat{G}$ (it is a Fell bundle).

Note that A_0 is a C*-subalgebra of A and there is a map $E(a) = \alpha_{e_0}(a) = \int_G \alpha_t(a)\, dt$ that projects onto it. E is a *conditional expectation* of A onto A_0:

1. $E : A \to A_0$, and if $a \in A_0$ then $E(a) = a$, so E is a projection.

2. If $a \geq 0$, then $E(a) \geq 0$.

3. If $a \in A, b \in A_0$, then $E(ba) = bE(a), E(ab) = E(a)b$ (so E is an (A_0, A_0)-bimodule homomorphism).

4. $\|E(a)\| \leq \|a\|$.

Cnsider $\ell^1(\mathbb{Z}^d)$ together with the cocycle

$$c_\theta(m, n) = e^{2\pi i m\cdot\theta n} \tag{130}$$

where $\theta \in M_d(\mathbb{R})$. This induces a twisted convolution and twisted *-structure. Let A_θ be the corresponding universal C*-algebra. For every $n \in \mathbb{Z}^d$ we have a corresponding indicator function which we will denote by U_n which satisfies

$$U_m U_n = c_\theta(m, n) U_{m+n}. \tag{131}$$

There is a dual action

$$\alpha_t(U_n) = \langle t, n\rangle U_n \tag{132}$$

where $t \in T^d$. Then there is a conditional expectation $E : A_\theta \to \mathbb{C}\mathrm{id}_{A_\theta}$ satisfying

$$E(U_m U_n) = \begin{cases} 0 \text{ if } m \neq -n \\ c_\theta(m, n) U_0 \text{ otherwise} \end{cases}. \tag{133}$$

Letting $E(a) = \tau(a)\mathrm{id}_{A_\theta}$, we have that τ is a faithful α-invariant trace. In fact it is the unique such trace.

Proof. If τ_0 is any other such trace, then

$$\tau_0(a)\mathrm{id}_{A_\theta} = \int \alpha_t(\tau_0(a))\, dt = \tau_0\left(\int \alpha_t(a)\right) = \tau_0\left(\tau(a)\mathrm{id}_{A_\theta}\right) = \tau(a) \tag{134}$$

and the conclusion follows. □

Proposition 7.2. *A_θ has no proper 2-sided α-invariant ideals.*

Proof. Let J be such an ideal. Then there exists $d \in J$ with $d \geq 0$ and $d \neq 0$. Furthermore, $\alpha_t(d) \in J$ for all t, hence $E(d) = \int \alpha_t(d)\, dt \in J$, so $\mathrm{id}_{A_\theta} \in J$ and J is not proper. □

Theorem 7.3. *The representation π of A_θ on $\ell^2(\mathbb{Z}^d)$ is faithful, so $C^*_r(\mathbb{Z}^d, c_\theta) = C^*(\mathbb{Z}^d, c_\theta)$.*

Proof. It suffices to show that the kernel of π is α-invariant. □

Let J be a 2-sided ideal in A_θ. Then for each $n \in \mathbb{Z}^d$ we have $U_n(J)U_n^* = J$. Now $U_n U_m U_n^* = c_{\theta-\theta^t}(n,m)U_m$; we define

$$\rho_\theta(n,m) = c_{\theta-\theta^t}(n,m). \tag{135}$$

The maps $\rho_\theta(n,-)$ can be identified with a subgroup of T^d. Let H_θ be the closure of this group. By the strong continuity of α we have $\alpha_t(J) = J$ for all $t \in H_\theta$.

Theorem 7.4. *If $H_\theta = T^d$, then A_θ is simple.*

In the case $d = 2$ consider U, V satisfying

$$VU = e^{2\pi i r}UV \tag{136}$$

where r is real. This corresponds to $C^*(\mathbb{Z}^2, c_\theta)$ where $\theta = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$. If r is irrational then this algebra is simple.

What can we say about the center of A_θ? We have

$$U_n U_m U_n^{-1} = \alpha_{\rho_\theta(n,m)} U_m \tag{137}$$

hence $U_m \in Z(A_\theta)$ if $\alpha_t(U_m) = U_m$ for all $t \in H_\theta$. In general a lies in the center iff $U_m a U_m^{-1} = a$ for all m, hence iff $\alpha_t(a) = a$ for all $t \in H_\theta$.

Let $D_\theta = \{m \in \mathbb{Z}^d : U_m \in Z(A_\theta)\}$, which is just $\{m \in \mathbb{Z}^d : \langle m, t\rangle = 1 \forall t \in H_\theta\}$, which we may also write as $H_\theta^\perp$. Let C_θ be the closed subalgebra of A_θ generated by the $U_m, m \in D_\theta$.

Theorem 7.5. $C_\theta = Z(A_\theta)$.

Proof. H_θ is a compact group, so we can equip it with normalized Haar measure. Define

$$Q : A_\theta \ni a \mapsto \int_{H_\theta} \alpha_t(a)\, dt \in A_\theta. \tag{138}$$

Then Q is a conditional expectation onto the center, and $Q(U_m) = U_m$ for all $m \in D_\theta$. For $m \notin D_\theta$, there exists $t_0 \in H_\theta$ such that $\langle m, t_0 \rangle \neq 1$, so

$$Q(U_m) = \int_{H_\theta} \alpha_t(U_m)\, dt = \int_{H_\theta} \langle m, t \rangle U_m \, dt = 0. \tag{139}$$

For any $f \in C_c(\mathbb{Z}^d)$ we therefore have $Q(f) \subseteq C_\theta$, hence $Q(A_\theta) \subseteq C_\theta$. □

We have $C_\theta \cong C^*(D_\theta) \cong C(\hat{D}_\theta) \cong C(\mathbb{T}^d/H_\theta)$, which is fairly explicit.

Let b be a Banach space and let α be a strongly continuous action of $\mathbb{R}$ on B. Given $b \in B$ we can ask whether the limit

$$\lim_{t \to 0} \frac{\alpha_t(b) - b}{t} \tag{140}$$

exists; if it does, we'll call it $D(b)$. More generally we can replace $\mathbb{R}$ with a finite-dimensional real vector space V. For $v \in V$ we can consider the action α_{tv} of $\mathbb{R}$ and ask whether the directional derivative

$$\lim_{t \to 0} \frac{\alpha_{tv}(b) - b}{t} \tag{141}$$

exists; if so, we'll call it $D_v(b)$.

Fact from Lie theory: every closed connected subgroup of $\mathrm{GL}_n(\mathbb{R})$ is a Lie group. These are the linear Lie groups. Every Lie group is locally isomorphic to a linear Lie group. In fact, for any Lie group G, either there is a discrete central subgroup C such that G/C is linear or there is a linear Lie group $\tilde{G}$ and a discrete central subgroup C of $\tilde{G}$ such that $\tilde{G}/C \cong G$.

Example $\mathrm{SL}_2(\mathbb{R})$ is linear but not simply connected. Its universal cover $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ is not linear.

Example The Heisenberg group

$$\left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \tag{142}$$

is linear, but its quotient by the discrete subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} \tag{143}$$

is not.

There is an exponential map

$$M_n(\mathbb{R}) \cong \mathfrak{gl}_n(\mathbb{R}) \ni X \mapsto \exp(X) \in \mathrm{GL}_n(\mathbb{R}) \tag{144}$$

and for any closed connected subgroup G of $\mathrm{GL}_n(\mathbb{R})$ we can set $\mathfrak{g}$ to be the collection of all $X \in \mathfrak{gl}_n(\mathbb{R})$ such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$. This is a Lie subalgebra. The exponential map $\mathfrak{g} \to G$ is a diffeomorphism in a neighborhood of 0. The subgroups of G that are locally isomorphic to $\mathbb{R}$ are exactly the subgroups $t \mapsto \exp(tX)$ for $X \in \mathfrak{g}$.

Now let G be a connected Lie group and let α be a strongly continuous action of G on B. For each $X \in \mathfrak{g}$ we can ask whether the limit

$$\lim_{t \to 0} \frac{\alpha_{\exp(tX)}(b) - b}{t} \tag{145}$$

exists, and if so we can denote it by $D_X(b)$. The collection of all b such that all iterated derivatives always exist is a linear subspace B^∞ of B, and on this subspace we have

$$(D_X D_Y - D_Y D_X)(b) = D_{[X,Y]}(b) \tag{146}$$

hence we have a representation of $\mathfrak{g}$.

Theorem 7.6. *(Gärding) For any $f \in C_c^\infty(G)$ and any $b \in B$, we have $\alpha_f(b) = B^\infty$. As a corollary, B^∞ is dense in B.*

Proof. We have

$$
\begin{aligned}
D_X(\alpha_f(b)) &= \lim_{t\to 0} \frac{\alpha_{\exp(tX)}(\alpha_f(b)) - \alpha_f(b)}{t} && (147)\\
&= \lim_{t\to 0} \frac{\alpha_{\exp(tX)} \int_G f(x)\alpha_x(b)\,dx - \int_G f(x)\alpha_x(b)\,dx}{t} && (148)\\
&= \lim_{t\to 0} \frac{1}{t}\left(\int f(x)\alpha_{\exp(tX)x}(b) - \int f(x)\alpha_x(b)\,dx\right) && (149)\\
&= \lim_{t\to 0} \int \frac{(\alpha_{\exp(tX)}f)(x) - f(x)}{t}\alpha_x(b)\,dx && (150)\\
&= \int (D_X f)(x)\alpha_x(b)\,dx && (151)
\end{aligned}
$$

where we use the fact that $(\alpha_y f)(x) = f(y^{-1}x)$. □

Is it true that every $b \in B^\infty$ has the form $\alpha_f(c)$ for some $f \in C_c^\infty(G)$ and $c \in B$? Dixmier-Malliavin showed that the answer is no, but that b is always a finite sum of terms of this form (so B^∞ is the *Gärding domain*).

The construction $b \mapsto \alpha_f(b)$ is often called smoothing or mollifying.

Example Let $G = \mathbb{T}^d, \mathfrak{g} = \mathbb{R}^d$. The exponential map here is the quotient map $\mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$. Let α be an action of $\mathbb{T}^d$ on B. Then we have isotypic components B_n, where $b_n \in B_n$ if

$$\alpha_v(b_n) = e^{2\pi i\langle n,v\rangle} b_n. \qquad (152)$$

For $b_n \in B_n$ we have $D_v(b_n) = 2\pi i\langle n, v\rangle b_n$. From here it follows that $B_n \subseteq B^\infty$. If $b \in B^\infty$ then

$$(D_X(b))_n = D_X(b_n) \qquad (153)$$

where b_n is the n^{th} isotypic component. Let $e_1, ...e_d$ be a basis of $\mathbb{R}^d$. Then

$$(D_{e_1}^{2p} D_{e_2}^{2p} ... D_{e_d}^{2p})(b_n) = (2\pi i)^{2pd}\langle n, e_1\rangle^{2p}\rangle ... \langle n, e_d\rangle^{2p} b_n \qquad (154)$$

from which it follows that

$$\|b_n\| \leq \frac{1}{1 + (\langle n, e_1\rangle ... \langle n, e_d\rangle)^p} \|(D_{e_1}^{2p} ... D_{e_d}^{2p})(b)\|. \qquad (155)$$

Hence if $b \in B^\infty$ then $\|b_n\|$ lies in the Schwartz space on $\mathbb{Z}^d$.

Proposition 7.7. *The converse holds. That is, if b_n is a function on $\mathbb{Z}^d$ with $b_n \in B_n$ such that $\|b_n\|$ lies in the Schwartz space, then $b = \sum b_n \in B^\infty$.*

Proof. Given $X \in \mathbb{R}^d$ regarded as the Lie algebra of $\mathbb{T}^d$ we have

$$\begin{aligned} D_X(b) &= \lim_{r\to 0} \frac{\alpha_{rX}(b) - b}{r} && (156)\\ &= \lim_{r\to 0} \sum_n \frac{\alpha_{rX}(b_n) - b_n}{r} && (157)\\ &= \lim_{r\to 0} \sum_n \frac{e^{2\pi i r\langle n, X\rangle} - 1}{r} b_n && (158)\\ &= \lim_{r\to 0} \sum_n 2\pi i \langle n, X\rangle b_n && (159)\end{aligned}$$

as expected (where everything converges appropriately because $\|b_n\|$ decays rapidly). □

8 Strict deformation quantizations

Let A be a C*-algebra and α an action of $G = \mathbb{T}^d$ on A. Then for $X \in \mathfrak{g}$ the derivative D_X is a derivation, so $D_X(ab) = D_X(a)b + aD_X(b)$. If θ is a skew-symmetric matrix, we can define a Poisson bracket

$$\{a, b\} = \sum \theta_{jk} D_{X_j}(a) D_{X_k}(b) \tag{160}$$

on A. Let (H, π, U) be a covariant representation of $(A, \mathbb{T}^d, \alpha)$ with π faithful. We have decompositions into isotypic components $A = \bigoplus A_n$ and $H = \bigoplus H_n$ with H_m, H_n orthogonal for distinct m, n and

$$U_t \pi(a_m) v_n = \langle t, m\rangle a_m \langle t, n\rangle v_n. \tag{161}$$

Hence $\pi(a_m)v_n \in H_{m+n}$.

Let $\theta \in M_d(\mathbb{R})$ and c_θ the corresponding cocycle. Define

$$\pi^\theta(a_m) v_n = c_\theta(m, n) \pi(a_m) v_n \tag{162}$$

and, given v with isotypic components v_n, define

$$\pi^\theta(a_m)v = \sum \pi^\theta(a_m)v_n. \tag{163}$$

Then $\|\pi^\theta(a_m)\| \leq \|a_m\|$. For $a \in A^\infty$, set $\pi^\theta(a) = \sum \pi^\theta(a_m)$. We have

$$\pi^\theta(a_m)\pi^\theta(b_n) = \pi^\theta(c(m,n)a_m b_n). \tag{164}$$

For $a, b \in A^\infty$, define

$$a *_\theta b = \sum_{m,n} c_\theta(m,n)a_m b_n. \tag{165}$$

Furthermore, define

$$a^{*_\theta} = \sum a_n^* c_\theta(n,n). \tag{166}$$

If θ is skew-adjoint then $c_\theta(n,n) = 1$, so the above reduces to the usual *-structure, and we can always pass to an equivalent θ with this property.

Let A_θ be the norm closure in $B(H)$ of $\{\pi_\theta(a) : a \in A^\infty\}$. Then $\mathbb{T}^d$ also acts on A_θ and $(A_\theta)^\infty = A^\infty$. The A_θ form a continuous field of C*-algebras over $M_d(\mathbb{R})$ with $A_0 = A$. For fixed θ we can think of $A_{\hbar\theta}, \hbar \in \mathbb{R}$ as a *deformation quantization* of A_0. In the *semiclassical limit* $\hbar \to 0$ we have

$$\|a *_\hbar b - ab\|_\hbar \to 0 \tag{167}$$

as $\hbar \to 0$. Moreover, a trace of the noncommutativity of the $A_{\hbar\theta}$ remains in a Poisson bracket on A_0 satisfying

$$\left\| \frac{a *_\hbar b - b *_\hbar a}{\hbar} - i\{a,b\} \right\|_\hbar \to 0 \tag{168}$$

as $\hbar \to 0$.

We can do the same given an action of $\mathbb{R}^d$ on a C*-algebra A. For $\mathbb{R}^d$ acting by translation on $\mathbb{R}^d$, the corresponding algebra A_θ is the Moyal quantization.

It is interesting to ask about when the A_θ are Morita equivalent. When $d = 2$ we have that if $\theta' = \frac{a\theta+b}{c\theta+d}$ where $\frac{ax+b}{cx+d} \in \mathrm{SL}_2(\mathbb{Z})$, then $A_\theta, A_{\theta'}$ are Morita equivalent. In higher dimensions, we can also have different smooth structures.

Given $(A, \mathbb{T}^d, \alpha)$ as above, letting Ω^k denote alternating linear k-forms on $\mathfrak{g}$ with values in A^∞, we can define $\Omega = \bigoplus \Omega^k$. This is equipped with a differential $d : \Omega^k \to \Omega^{k+1}$ and it is graded, but not graded-commutative.

$\mathbb{T}^d \subset \mathrm{SO}(n+1)$ acts on S^n, and we can use this action to get a quantized version of S^n. Now, $\mathbb{T}^d$ will not act on the quantized version, but a quantized version of the group does.

8.1 K-theory

Definition A *vector bundle* over a topological space X is a space E and a surjection $\pi : E \to X$ such that each fiber $\pi^{-1}(x)$ has the structure of a finite-dimensional real vector space and such that each point $x \in X$ has a neighborhood $U \ni x$ such that $\pi^{-1}(U) \cong U \times \mathbb{R}^d$ (in a way that respects both the projection maps and the linear structure).

We would like to convert this definition into algebraic language to see what a vector bundle over a noncommutative space should look like. Given a vector bundle E, let $\Gamma(E)$ be the space of (continuous) sections of the projection map π. This is a $C(X)$-module.

Assume X is compact. Choose an open cover U_i such that E is trivial over each U_i. By compactness it has a finite subcover. On each trivialization take the standard inner product. Let φ_j be a partition of unity subordinate to the cover. For each j set

$$\langle v, w\rangle_j(x) = \varphi_j(x)\langle v(x), w(x)\rangle \tag{169}$$

where $v, w \in \Gamma(E)$, then set

$$\langle v, w\rangle(x) = \sum_j \langle v, w\rangle_j(x). \tag{170}$$

This inner product has the following properties:

1. $\langle v, w\rangle^* = \langle w, v\rangle$
2. $\langle v, hw\rangle = \langle v, w\rangle h$
3. $\langle v, v\rangle = 0 \Rightarrow v = 0$.

Using the inner product we can define rank-1 operators (roughly speaking) $\langle v, w\rangle_0 u = u\langle w, v\rangle_A$.

For each $x \in X$ we can choose in each $\pi^{-1}(x)$ an orthonormal basis $e_1, ...e_d$, and we can find $h_x^1, ...h_x^k \in \Gamma(E)$ such that $h_x^k(x) = e_k$, hence $\sum \langle h_x^k, h_x^k \rangle_0(x) = \mathrm{id}_{\pi^{-1}(x)}$. So there is a neighborhood U_x of x such that $\sum \langle h_x^k, h_x^k \rangle_0(y) \geq \frac{1}{2}\mathrm{id}_{\pi^{-1}(y)}$ by continuity for $y \in U_x$. By compactness, the U_x have a finite subcover $U_{x_1}, ...U_{x_\ell}$. Then

$$S = \sum_{k,\ell} \langle h_{x_\ell}^k, h_{x_\ell}^k \rangle_0 \geq \frac{1}{2}\mathrm{id}. \tag{171}$$

Generally, let A be a *-subalgebra of a C*-algebra. Let M be a right A-module with an A-valued inner product $\langle -, - \rangle_A$. Then a set h_i of elements of M such that

$$\sum \langle h_i, h_i \rangle_0 = \mathrm{id} \tag{172}$$

is called a standard *module frame.*

Theorem 8.1. *Let A be a *-subalgebra of a C*-algebra, M a right A-module with an A-valued inner product and standard module frame h_i. Then M is a projective A-module and M is self-dual with respect to the inner product.*

Corollary 8.2. *(Swan) $\Gamma(E)$ is a projective $C(X)$-module.*

Proof. Define $\Phi : M \to A^n$ by

$$(\Phi(v))_j = \langle h_j, v \rangle_A. \tag{173}$$

Φ is an injective A-module homomorphism. Define $P \in M_n(A)$ by $P_{jk} = \langle h_j, h_k \rangle_A$, so that

$$(P^2)_{ij} = \sum_k \langle h_i, h_k \rangle_A \langle h_k, h_j \rangle_A = \langle h_i, \sum h_k \langle h_k, h_j \rangle_A \rangle = \langle h_i, h_j \rangle = P_{ij} \tag{174}$$

and $P^* = P$, so P is a self-adjoint idempotent. Its range is the range of Φ, which is isomorphic to M. Hence M is projective.

Suppose $F \in \mathrm{Hom}_A(M, A_A)$. Then for $v \in M$,

$$v = \sum h_j \langle h_j, v \rangle_A \tag{175}$$

and

$$F(v) = \sum F(h_j)\langle h_j, v\rangle_A = \sum \langle h_j F(h_j)^*, v\rangle_A. \tag{176}$$

so F is the inner product with $\sum h_j F(h_j)^*$ as desired. $\square$

Exercise 8.3. *Let $C(T)$ denote 1-periodic functions on $\mathbb{R}$. Consider the modules*

$$M_n^{\pm} = \{f : \mathbb{R} \to \mathbb{R} : f(t+n) = \pm f(t)\}. \tag{177}$$

Show that these are projective modules. Which of them are free?

Let $P : X \to M_n(\mathbb{R})$ be continuous where $P(x)$ is always a projection, and set the fiber at $x \in X$ to be the range of $P(x)$. Then this defines a vector bundle. More generally, for any unital ring R, all finitely-generated projective R-modules are direct summands of free modules, so can be obtained as the image of projections in $M_n(R)$. If V, W are f.g. projective, so is $V \oplus W$. This gives an addition on the isomorphism classes of f.g. projective modules. The resulting semigroup is not necessarily cancellative.

Example Let $X = S^2$ and consider real vector bundles. Let V be the tangent bundle and N the normal bundle. Then $V \oplus N \cong \mathbb{R}^3 \cong \mathbb{R}^2 \oplus N$ (since N is trivial) but $V \not\cong \mathbb{R}^2$.

To force cancellation, we take the Grothendieck group. For a ring R this group is denoted $K_0(R)$. We can think of $K_0(R)$ as a set of equivalence classes of projections in matrix rings over R. This is a functor.

Let R be a k-algebra, k a field, and $\varphi : R \to R/I$ a surjective homomorphism of k-algebras with kernel I. Then the short exact sequence $I \to R \to R/I$ gives rise to an exact sequence

$$K_0(I) \to K_0(R) \to K_0(R/I) \tag{178}$$

where $K_0(I)$ is defined as follows: adjoin a unit to get a ring $\tilde{I}$, which has some K-theory $K_0(\tilde{I})$. There is a map $K_0(\tilde{I}) \to K_0(k) \cong \mathbb{Z}$, and we define $K_0(I)$ to be its kernel. The above extends to an exact sequence

$$K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I) \tag{179}$$

where $K_1(R)$ is defined as follows: given R, consider the filtered colimit $\mathrm{GL}_\infty(R)$ of the groups $\mathrm{GL}_n(R)$, and abelianize it. This is also a functor. If R is a unital Banach algebra, then we instead quotient by the connected component of the identity.

We defined K_0 and an algebraic and topological version of K_1. There is also an algebraic K_2, and Quillen defined K_n for all n satisfying $K_n^{\text{top}}(A) \cong K_{n-1}^{\text{top}}(SA)$ where SA is the suspension $C_0((0,1), A)$. *Bott periodicity* states that over $\mathbb{C}$, K_n has period 2, and over $\mathbb{R}$, it has period 8.

Let A_θ be the rotation algebra $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, so that

$$(\alpha_n(f))(x) = f(x - n\theta) \tag{180}$$

where $f \in C(\mathbb{T}), n \in \mathbb{Z}, x \in \mathbb{T}$. For awhile it was an open question whether there were projections in such an algebra. It turns out that they exist if $0 < \theta < 1$.

Let $\theta \in \mathbb{R}^+$ and let $\Xi = C_c(\mathbb{R})$. Let $C(\mathbb{T})$ act by pointwise multiplication and let δ_i act by

$$(\delta_n \xi)(t) = \xi(t - n\theta). \tag{181}$$

What is the commutant of this action? It includes multiplication by the functions on $\mathbb{R}$ of period θ as well as translation by the integers. This in fact generates an algebra isomorphic to $A_{1/\theta}$, and the two are Morita equivalent.

Let $\delta_0 \in \mathbb{Z}$ denote the identity. Let $M_f = f\delta_0$ and let $U = M_1\delta_1$. We'll look for projections of the form

$$P = M_h U^{-1} + M_f + M_g U. \tag{182}$$

Taking the adjoint gives

$$P^* = M_{\alpha(\bar{h})} U + M_{\bar{f}} + M_{\alpha^{-1}(\bar{g})} U^{-1} \tag{183}$$

so f is real-valued and $g = \alpha(\bar{h})$. Squaring gives $g\alpha(g) = 0$ and $g = fg + g\alpha(f)$, hence $g(1 - f - \alpha(f)) = 0$, and

$$f = f^2 + h\alpha^{-1}(g) = g\alpha(h) = f^2 + \alpha^{-1}(g\bar{g}) + g\bar{g} \geq 0. \tag{184}$$

If $\theta < 1$, choose $\epsilon > 0$ such that $\theta > \epsilon < 1$. Then we can take f which vanishes outside of $[0, \theta + \epsilon]$ and which is equal to 1 on $[\epsilon, \theta]$, and we can take $g = \sqrt{f - f^2}$.

A_θ has a canonical tracial state which is the one invariant under the action of the Pontrjagin dual: it sends $F(t,n)$ to $\int_T F(t,0)\,dt$. If τ denotes this state, we have $\tau(P) = \int_T f\,dt = \theta$.

We can compute that $K_0(A_\theta) = \mathbb{Z}^2$.

Given a C*-algebra A with identity and a trace τ on A, we get a trace on $M_n \otimes A$ such that $\tau(UPU^{-1}) = \tau(P)$ if U is unitary. If $t \mapsto P_t$ is a continuous path of projections, then $\tau(P_t)$ induces a map $K_0(A) \to \mathbb{R}$.

Let $\Xi(n,q)$ denote functions $f : \mathbb{R}^2 \to \mathbb{C}$ such that $f(s+1,t) = f(s,t)$ and $f(s,t+n) = e^{2\pi isq} f(s,t)$. These are all projective modules over $C(\mathbb{T}^2)$.

Cancellation holds for projective modules over A_θ but not in general.