This is a practice quiz!

Name and section: ______________________________

1. Let f be a function. Define carefully: $\lim_{x\to\infty} f(x) = \infty$ if and only if...

   Solution: ...for all $N > 0$ there exists $M > 0$ such that if $x > M$ then $f(x) > N$.

2. Let $f(x) = \arctan(\ln x)$.
   (a) Explain why f is one-to-one.
   (b) What is the domain of f? What is the domain of f^{-1}?
   (c) Find a formula for f^{-1}.

   Solution:

   (a) f is one-to-one if and only if $f(x) = f(y)$ implies $x = y$. If $\arctan(\ln x) = \arctan(\ln y)$, then by taking the tangent of both sides we get $\ln x = \ln y$, and by taking the exponential of both sides we get $x = y$.

   Alternatively, arctan is an increasing function and so is $\ln x$, so their composition is increasing too, and increasing functions are always one-to-one.

   (b) Since arctan has domain all of $\mathbb{R}$, $\arctan \ln x$ is undefined if and only if $\ln x$ is undefined. So the domain of f is the same as the domain of $\ln x$, or the positive reals $\boxed{(0, \infty)}$.

   The domain of f^{-1} is the same as the range of f. Since $\ln x$ takes on all real values, $\arctan \ln x$ takes on all the values that $\arctan x$ does. So the range of f is the same as the range of $\arctan x$, or the interval $\boxed{(-\frac{\pi}{2}, \frac{\pi}{2})}$.

   (c) We can do this by writing $x = \arctan \ln y$ and solving for y. This gives $\tan x = \ln y$ and hence

$$y = f^{-1}(x) = \boxed{e^{\tan x}}. \tag{1}$$

   (But note that we have to restrict the domain of this function to $(-\frac{\pi}{2}, \frac{\pi}{2})$ for it to really be the inverse. The function $e^{\tan x}$ has a larger domain than f^{-1} does.)

3. Compute the following limits (be specific about infinite limits).

(a) $\lim_{x \to 0} \frac{\tan 2x}{x}$

(b) $\lim_{x \to -\infty} \frac{x^3+1}{\sqrt{x^6+6x}}$

(c) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+x}-x}$

Solution:

(a) We can write $\tan 2x = \frac{\sin 2x}{\cos 2x}$, so that the limit is

$$\lim_{x \to 0} \frac{\sin 2x}{x \cos 2x}. \tag{2}$$

Next, recalling that $\lim_{x \to 0} \frac{\sin x}{x}$, we can rewrite this limit as

$$\lim_{x \to 0} 2 \frac{\sin 2x}{2x} \frac{1}{\cos 2x}. \tag{3}$$

Using the fact that the product of limits is the limit of the products, the fact that cos is continuous, and the fact that taking $x \to 0$ is the same as taking $2x \to 0$, we conclude that the limit is

$$2 \left(\lim_{x \to 0} \frac{\sin 2x}{2x} \right) \left(\lim_{x \to 0} \frac{1}{\cos 2x} \right) = 2(1)(1) = \boxed{2}. \tag{4}$$

(b) Dividing the numerator and denominator by x^3 gives

$$\lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{\frac{\sqrt{x^6+6x}}{x^3}}. \tag{5}$$

We would like to absorb x^3 into the square root, but since $x \to -\infty$, x^3 will eventually be negative, so we need to introduce a negative sign to do this. The limit becomes

$$\lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{-\frac{\sqrt{x^6+6x}}{\sqrt{x^6}}} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x^3}}{-\sqrt{1 + \frac{6}{x^5}}}. \tag{6}$$

But now the numerator has limit 1 and the denominator has limit -1, so using the fact that the quotient of limits is the limit of the quotients (if the denominator has a nonzero limit) we conclude that the original limit is $\boxed{-1}$.

(c) Multiplying the numerator and denominator by the conjugate gives

$$\lim_{x \to \infty} \frac{x(\sqrt{x^2+x}+x)}{(x^2+x)-x^2} = \lim_{x \to \infty} \frac{x(\sqrt{x^2+x}+x)}{x} = \lim_{x \to \infty} (\sqrt{x^2+x}+x). \tag{7}$$

This is a sum of two terms going to $+\infty$, so goes to $\boxed{+\infty}$.

4. Let $f(x), g(x)$ be two functions such that $g(1) = 2, f(1) = 3, g'(1) = 4, f'(1) = 5$. Compute the derivatives of the following functions at 1.

   (a) $f(x) + g(x)$

   (b) $f(x)g(x)$

   (c) $\frac{f(x)}{g(x)}$

   Solution:

   (a) By the sum rule the derivative is $f'(1) + g'(1) = 4 + 5 = \boxed{9}$.

   (b) By the product rule the derivative is $f'(1)g(1) + f(1)g'(1) = 5 \cdot 2 + 3 \cdot 4 = \boxed{22}$.

   (c) By the quotient rule the derivative is $\frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{5 \cdot 2 - 3 \cdot 4}{2^2} = \boxed{-\frac{1}{2}}$.