> *"Forget perfect on the first try. In the face of frustration, your best tool is a few deep breaths, and remembering that you can do anything once you've practiced two hundred times."*

Name and section: ______________________________

1. Consider the solid obtained by rotating the region R bounded by $y = x^3, y = 0, x = 1$ about $x = 2$.

   (a) Sketch the region R and the solid.

   Solution: You're on your own! Check with a graphing calculator.

   (b) Compute the volume of the solid using washers.

   Solution: To use washers we'll have to slice R horizontally, so the slices are indexed by y. The inner radius of the washer at y is always 1 and the outer radius is $2 - \sqrt[3]{y}$, so we get

$$\begin{aligned} V &= \int_0^1 \pi \left((2 - \sqrt[3]{[y]})^2 - 1^2 \right) dy && (1)\\ &= \pi \int_0^1 \left(y^{2/3} - 2y^{1/3} + 3 \right) dy && (2)\\ &= \pi \left(\frac{3}{5} y^{5/3} - 4\frac{3}{4} y^{4/3} + 3y \right) \Big|_0^1 && (3)\\ &= \pi \left(\frac{3}{5} - 3 + 3 \right) && (4)\\ &= \frac{3\pi}{5}. && (5) \end{aligned}$$

   (c) Compute the volume of the solid using shells.

   Solution: To use shells we'll have to slice R vertically, so the slices are indexed by x. The radius of the shell at x is always $2 - x$ and the height is x^3, so we get

$$\begin{aligned} V &= \int_0^1 2\pi(2 - x)x^3\, dx && (6)\\ &= 2\pi \int_0^1 (2x^3 - x^4)\, dx && (7)\\ &= 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \right) \Big|_0^1 && (8)\\ &= 2\pi \left(\frac{1}{2} - \frac{1}{5} \right) && (9)\\ &= \frac{3\pi}{5}. && (10) \end{aligned}$$

2. Consider the function $f(x) = x^3 - x$.

   (a) Find the intervals of increase and decrease of f.

   Solution: $f'(x) = 3x^2 - 1$, so the critical points are $x = \pm\frac{1}{\sqrt{3}}$. The derivative is positive (so f is increasing) when $x > \frac{1}{\sqrt{3}}$ or $x < -\frac{1}{\sqrt{3}}$ and negative (so f is decreasing) when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

   (b) Find the local maxima and minima of f.

   Solution: From the above calculations, it follows by the first derivative test that $x = \frac{1}{\sqrt{3}}$ is a local minimum and $x = -\frac{1}{\sqrt{3}}$ is a local maximum.

   (c) Find the intervals of concavity and the inflection points.

   Solution: $f''(x) = 6x$, so the only inflection point is $x = 0$. The second derivative is positive (so f is concave up) when $x > 0$ and negative (so f is concave down) when $x < 0$.

   (d) Sketch a graph of f.

   Solution: Graphing calculator!

3. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution: The ladder, together with the wall and the floor, forms the hypotenuse of a right triangle. Let $x(t), y(t), z(t)$ denote the lengths of the sides of this triangle at a time t (with x the side parallel to the x-axis, y the side parallel to the y-axis, and z the hypotenuse / ladder). By the Pythagorean theorem, we know that

$$x^2 + y^2 = z^2. \tag{11}$$

Differentiating (since this is a related rates problem!) we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}. \tag{12}$$

Now we use the information given in the problem. First, we're given that $z = 10$, and in particular z is constant, so $\frac{dz}{dt} = 0$. Second, we're given that $\frac{dx}{dt} = 1$ all the time. Finally, at a particular time t_0 we're told that $x(t_0) = 6$ and asked to find $\frac{dy}{dt}|_{t=t_0}$. Altogether, this gives

$$6^2 + y(t_0)^2 = 10^2 \tag{13}$$

$$2 \cdot 6 \cdot 1 + 2y(t_0)\frac{dy}{dt}\bigg|_{t=t_0} = 0. \tag{14}$$

Hence $y(t_0) = 8$ and

$$\frac{dy}{dt}\bigg|_{t=t_0} = -\frac{2 \cdot 6}{2 \cdot 8} = -\frac{3}{4}. \tag{15}$$

4. (a) State the precise definition of the derivative $f'(a)$.

Solution: $f'(a)$ is the value of the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{16}$$

or equivalently it is the value of the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{17}$$

The two limits are related by the substitution $x = a + h$.

(b) Directly from the definition of a derivative, show that if $f(x) = x^2$ then $f'(2) = 4$.

Solution: We will show more generally that $f'(a) = 2a$. To see this using the first definition, use difference of squares:

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^2 - a^2}{x - a} && (18)\\
&= \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} && (19)\\
&= \lim_{x \to a} (x + a) && (20)\\
&= 2a. && (21)
\end{aligned}$$

To see this using the second definition, just expand and cancel as appropriate:

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} && (22)\\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} && (23)\\
&= \lim_{h \to 0} \frac{2ah + h^2}{h} && (24)\\
&= \lim_{h \to 0} (2a + h) && (25)\\
&= 2a. && (26)
\end{aligned}$$