Make Your Work Easy to Check

Qiaochu Yuan

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In section today we looked at the following problem.

Example (Stewart, 3.4.74) Let $F(x) = f(xf(xf(x)))$, where f is a function satisfying $f(1) = 2, f(2) = 3, f'(1) = 4, f'(2) = 5, f'(3) = 6$. Compute $F'(1)$.

This problem is complicated enough that it would be annoying to do all of the necessary differentiations in your head. If you did, your work might look like this:

$$\begin{aligned} F'(x) &= f'(xf(xf(x)))\,(x\,(f'(xf(x))\,(xf'(x) + f(x))) + f(xf(x))) \\ F'(1) &= f'(1f(1f(1)))\,((f'(1f(1))(f(1) + xf'(1))) + f(1f(1))) \\ &= 6\,(5\,(4 + 2) + 3) = 198. \end{aligned}$$

Is that even right? Did I miss a term in there somewhere? Either way, it's hard to tell. The way I did my work above condenses a lot of steps into one and makes it **very hard to check**. That's a bit unfortunate.

Let's try separating out all of the steps:

$$\begin{aligned} F'(x) &= f'(xf(xf(x)))\,(xf(xf(x)))' && \text{(chain rule)} \\ &= f'(xf(xf(x)))\,\big(x\,(f(xf(x)))' + f(xf(x))\big) && \text{(product rule)} \\ &= f'(xf(xf(x)))\,\big(x\,(f'(xf(x))\,(xf(x)))' + f(xf(x))\big) && \text{(chain rule)} \\ &= f'(xf(xf(x)))\,(x\,(f'(xf(x))\,(xf'(x) + f(x))) + f(xf(x))) && \text{(product rule).} \end{aligned}$$

That's a little better - now it's clear that I need to check four things, two uses of the chain rule and two uses of the product rule - but everything's still squished together and by the time I'm checking the last step I still have to wade through an awful lot of parentheses. My work is **still hard to check**.

I can make my life a little easier by substituting $x = 1$ partway through once I've encountered a term I don't need to differentiate anymore. I won't get equalities anymore, since I need to distinguish $F'(x)$ from $F'(1)$, so I'll use arrows instead of equality signs:

$$
\begin{aligned}
F'(1) &\to f'(1f(1f(1)))\,(xf(xf(x)))' && \text{(chain rule)}\\
&\to 6\,\big(1\,(f(xf(x)))' + f(1f(1))\big) && \text{(product rule)}\\
&\to 6\,\big((f'(1f(1))\,(xf(x)))' + 3\big) && \text{(chain rule)}\\
&\to 6\,(5\,(1f'(1) + f(1)) + 3) && \text{(product rule)}\\
&\to 6\,(5\,(4+2)+3) = 198.
\end{aligned}
$$

This is even better! I don't have to carry around those pesky terms like $f'(xf(xf(x)))$ that evaluate to nice numbers later anyway if I just evaluate them once I can. One remaining issue here is that, as written, I need to simultaneously check my differentiation rules and that I've correctly substituted $x = 1$. In general it's better to separate out such steps to make them even easier to check.

An alternative to attacking the problem the way I've been attacking it, which is from the "outside in," is to try to attack them from the "inside out." That is, I know at some point that in the course of differentiating $F(x)$ I'm going to have to differentiate all of the functions appearing inside of it, so I might as well do them first. So, let's start from the inside and work out: the innermost function has derivative

$$\frac{d}{dx}xf(x) = xf'(x) + f(x) \to 1f'(1) + f(1) = 6$$

by the product rule, where again I'm using an arrow $\to$ to denote substituting 1 where I can. The next function has derivative

$$\frac{d}{dx}f(xf(x)) \to f'(xf(x)) \cdot 6 \to f'(1f(1)) \cdot 6 = 30$$

by the chain rule, where I'm immediately writing down 6 instead of the value of $(xf(x))'$ at 1 because I've already calculated this in the first step. The next function has derivative

$$\frac{d}{dx}xf(xf(x)) \to x \cdot 30 + f(xf(x)) \to 30 + f(1f(1)) = 33$$

by the product rule, where again I'm immediately writing down 30 instead of the value of $(f(xf(x)))'$ at 1 because I already calculated this in the previous step. Finally, the last function, which is the original function we wanted to differentiate, is

$$\frac{d}{dx}f(xf(xf(x))) \to f'(xf(xf(x))) \cdot 33 \to f'(1f(1f(1))) \cdot 33 = 198$$

by the chain rule.

One way to describe what we just did is that we **thought ahead**: in the other two approaches, in order to compute $F'(1)$ we ended up having to compute various other derivatives. In this approach we anticipated what those derivatives were and computed them first to make our lives easier. Best of all, the way we did this made each step a fairly straightforward (both to perform and to check!) application of a single differentiation rule.

Several of the other problems in this section of Stewart should be easier using the method(s) described above. Try them out!

Example (Stewart, 3.4.45) Find the derivative of $\cos\sqrt{\sin(\tan\pi x)}$.

Example (Stewart, 3.4.71) Let $r(x) = f(g(h(x)))$, where $h(1) = 2, g(2) = 3, h'(1) = 4, g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

Example (Stewart, 3.4.72) If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

Example (Stewart, 3.4.73) If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.