Integrals $\iint$, $\iiint$ $\quad \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \to \int_V$

Riemann integral (Lebesgue integral)

$0\le x\le 1 \qquad f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$

$$\boxed{\iint_D f(x,y)\,dA = 0 \qquad \text{Area } D = 0}$$

1d) $f(x)$, $a\le x\le b$ $\quad \int_a^b f(x)dx \quad$ Continuous

P - partition of $[a,b]$: $a=t_0\le t_1\le\dots\le t_{n-1}\le t_n=b$

mesh $P = \max_i |t_{i+1}-t_i|$. If all intervals are the same then mesh $P=\frac{b-a}{n}$

$$S_n=\sum_{i=0}^{n-1} f\left(\frac{t_{i+1}-t_i}{2}\right)|t_{i+1}-t_i| \quad \text{- Riemann sum}$$

$$\lim_{n\to\infty} S_n = \int_a^b f(x)dx$$

$$\boxed{\iint_D 1 = \text{Area}(D)}$$

$m\le f\le M$: $\quad m\,A(D)\le \iint_D f\le M\,A(D)$

2d Riemann integral

$z=f(x,y)$ $\quad (x,y)\in R=[a,b]\times[c,d]$

Partition $P(R)=P([a,b])\times P([c,d])$ $\quad$ mesh R = mesh $P([a,b])\times$ mesh $P([c,d])$

$P([a,b])$: $a=t_0\le t_1\le\dots\le t_{n-1}\le t_n=b$ $\quad$ mesh $P([a,b])=\max_i|t_{i+1}-t_i|$

$P([c,d])$: $c=s_0\le s_1\le\dots\le s_{m-1}\le s_m=d$ $\quad$ mesh $P([c,d])=\max_j|s_{j+1}-s_j|$

(t_i,s_j), $i=1,\dots,n$, $j=1,\dots,m$ $\quad P_{ij}\ni(x_{ij}^*,y_{ij}^*)$ $\quad \Delta A_{ij}$ - area of R_{ij}

$$S_{n,m}=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\overbrace{f(x_{ij}^*,y_{ij}^*)\underbrace{|t_{i+1}-t_i||s_{j+1}-s_j|}_{\Delta A_{ij}}}^{\Delta V_{ij}} \quad \text{- Riemann sum}$$

$$\lim_{\substack{\text{mesh } R\to0\\ m,n\to\infty}} S_{n,m}=\int_a^b\int_c^d f(x,y)\,dxdy=\iint_R f(x,y)\,dA=\iint_R f$$

differential form

$\iint_R f$ = Volume of the 3d solid under the surface $z=f(x,y)$, $(x,y)\in R$, $f(x,y)\ge0$

Ex R: $-1\le x\le1$, $-2\le y\le2$ $\quad f(x,y)=\sqrt{1-x^2}$

$$\text{Volume}=\int_{-1}^{1}\int_{-2}^{2}\sqrt{1-x^2}\,dx\,dy=\left(\int_{-2}^{2}dy\right)\left(\int_{-1}^{1}\sqrt{1-x^2}dx\right)=4\cdot\frac{\pi}{2}=2\pi$$

Iterated integrals

x^5y^7, $(x,y)\in R$: $a\le x\le b$, $c\le y\le d$

$$\iint_R x^5y^7dxdy=\int_a^b\int_c^d x^5y^7dxdy=\int_c^d y^7\left(\int_a^b x^5dx\right)dy=\int_c^d y^7\frac{b^6-a^6}{6}dy=\frac{(d^8-c^8)(b^6-a^6)}{48}$$

$$\left(\int_a^b x^5dx\right)\left(\int_c^d y^7dy\right)$$

Theorem (Fubini) If f is continuous on $R=[a,b]\times[c,d]$ (allow disc. in a finite number of points, + bounded) then

$$\iint_R f=\int_a^b\int_c^d f(x,y)dxdy=\int_c^d\int_a^b f(x,y)dydx$$

$$\int_c^d dy\int_a^b f(x,y)dx = \int_a^b dx\int_c^d f(x,y)dy$$

Ex $\iint_R y\sin(xy)\,dx\,dy$ $\quad R=[1,2]\times[0,\pi]$

$$=\int_0^\pi dy\int_1^2 y\sin xy\,dx=\int_0^\pi dy\int_1^2\sin(xy)\,d(xy)=\int_0^\pi dy(-\cos2y+\cos y)=-\frac12\sin2y\Big|_0^\pi+\sin y\Big|_0^\pi=0$$

Ex
$$\int_0^2\int_1^2(x-3y^2)dxdy=\int_0^2dx\int_1^2(x-3y^2)dy=\int_0^2dx\left(xy\Big|_1^2-y^3\Big|_1^2\right)=\int_0^2dx(2x-x-8+1)=\int_0^2dx(x-7)=\frac{x^2}{2}-7x\Big|_0^2=\frac42-14=-12$$
$$=\int_1^2dy\int_0^2(x-3y^2)dx=\int_1^2dy\left(\frac{x^2}{2}-3xy^2\right)\Big|_0^2=\int_1^2dy\left(\frac42-6y^2\right)=\int_1^2dy(2-6y^2)=2y-2y^3\Big|_1^2=4-16-2+2=-12$$

Average value 1d) $\langle f\rangle=\frac{1}{b-a}\int_a^b f(x)dx$ $\quad$ 2d) $\langle f\rangle=\frac{1}{\text{Area } R}\iint_R f(x,y)dxdy$

Integration over arbitrary regions (15.2)

$\iint_D f(x,y)\,dxdy$ $\quad D\subset\mathbb{R}^2$ $\qquad D\subset R\subset\mathbb{R}^2$

Define $f(x,y)$ on $R\supset D$: $f(x,y)=\begin{cases}f(x,y), & x\in D\\ 0, & x\in R\setminus D\end{cases}$ $\quad \iint_D f=\iint_R f$

Type II $\quad$ Type I

$f(x,y)$, $(x,y)\in D$ $\quad \{(x,y)\in\mathbb{R}^2 \mid x_1\le x\le x_2,\ g(x)\le y\le h(x)\}$

$$\boxed{\iint_D f(x,y)dxdy=\int_a^b dx\int_{h(x)}^{g(x)}dy\,f(x,y)}$$

$$\iint_D f(x,y)=\int_c^d dy\int_{\alpha(y)}^{\beta(y)}dy\,f(x,y)$$

Ex D: $y=2x^2$, $y=1+x^2$
$$\iint_D(x+2y)\,dA=\int_{-1}^{1}dx\int_{2x^2}^{1+x^2}(x+2y)dy$$
Intersection points $2x^2=1+x^2$, $x^2=1$, $x=\pm1$
$$=\int_{-1}^1dx\,(xy+y^2)\Big|_{2x^2}^{1+x^2}=\int_{-1}^1dx\left(x(1+x^2-2x^2)+(1+x^2)^2-4x^4\right)=\int_{-1}^1dx(x-x^3+1+2x^2-3x^4)$$
$$=\left(\frac{x^2}{2}-\frac{x^4}{4}+x+\frac23x^3-\frac35x^5\right)\Big|_{-1}^1=2+2\cdot\frac23-2\cdot\frac35=2\,\frac{15+10-9}{15}=\frac{32}{15}$$

Ex D: $x=\frac12y^2-3$, $x=y+1$
$$\iint_D xy\,dxdy=\int_{-2}^4dy\int_{\frac{y^2}{2}-3}^{y+1}xy\,dx$$
Intersection pts $\frac12y^2-3=y+1$, $y^2-2y-8=0$, $y=\{-2,4\}$
$$=\int_{-2}^4dy\left(\frac{x^2y}{2}\right)\Big|_{\frac{y^2}{2}-3}^{y+1}=\frac12\int_{-2}^4dy\left(y(y+1)^2-\left(\frac{y^2}{2}-3\right)^2y\right)=\frac12\int_{-2}^4\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy=\frac12\left[-\frac{y^6}{24}+y^4+\frac23y^3-4y^2\right]_{-2}^4=36$$

Ex $z=2-x-2y$, $x+2y+z=2$, $x=2y$; $(0,0,2)$, $(0,1,0)$, $(1,\frac12,0)$
$$Vol=\iint_T(2-x-2y)\,dxdy=\int_0^1dx\int_{x/2}^{1-x/2}(2-x-2y)dy=\int_0^1dx\left(2y-xy-y^2\right)\Big|_{x/2}^{1-x/2}$$
$$=\int_0^1dx\left(2-x-x-x\left(1-\frac x2\right)+\frac{x^2}{2}-\left(1-\frac x2\right)^2+\left(\frac x2\right)^2\right)$$
$x+2y=2 \Rightarrow y=1-\frac x2$; $\quad x=2y\Rightarrow y=\frac x2$
$$=\int_0^1dx(1-2x+x^2)=\int_{-1}^0 d(x-1)\,(x-1)^2=\int_{-1}^0du\,u^2=\frac{u^3}{3}\Big|_{-1}^0=0-\left(-\frac13\right)=\frac13$$

Properties of double integrals

- $\iint_D(\alpha f(x,y)+\beta g(x,y))=\alpha\iint_D f(x,y)+\beta\iint_D g(x,y)$
- If $f(x,y)\ge g(x,y)$ $\forall(x,y)\in D$ then $\iint_D f(x,y)\ge\iint_D g(x,y)$
- Additivity $D=D_1\sqcup D_2$, $D_1\cap D_2=\emptyset$ then $\iint_D f=\iint_{D_1}f+\iint_{D_2}f$

Area D $\iint_D 1\,dxdy=\text{Area}(D)$

Lemma Let $m\le f(x,y)\le M$ $\forall(x,y)\in D$ then $\text{Area}(D)\,m\le\iint_D f(x,y)dxdy\le\text{Area}(D)\,M$

(15.3) Double integrals in polar coordinates

$x=r\cos\theta$, $y=r\sin\theta$

$$\iint_D f(x,y)\,dxdy\to\iint_D f(r\cos\theta,r\sin\theta)\,r\,d\theta dr$$
$f(x,y)=f(r\cos\theta,r\sin\theta)$
$dA=dxdy\to r\,dr\,d\theta$

Partition of $[\theta_1,\theta_2]$, $[r_1,r_2]$: θ,r $\qquad \theta_1\le\theta\le\theta_2$, $r_1\le r\le r_2$

$$\Delta A=\frac12(r+\Delta r)^2\Delta\theta-\frac12r^2\Delta\theta=\frac12\Delta\theta\left(\cancel{r^2}+2r\Delta r+(\Delta r)^2-\cancel{r^2}\right)=r\Delta\theta\Delta r+\frac12\cancel{\Delta\theta(\Delta r)^2}$$
$\Delta\theta,\Delta r\to0$ $\qquad \Delta A\to dA=r\,dr\,d\theta$

Ex $A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le2\}$ $\quad h_1(\theta)$, $h_2(\theta)$
$$\iint_A(3x+4y^2)dxdy=\int_0^{2\pi}\int_1^2(3r\cos\theta+4r^2\sin^2\theta)\,r\,d\theta\,dr$$
$$=\int_0^{2\pi}d\theta\int_1^2dr(3r^2\cos\theta+4r^3\sin^2\theta)=\int_0^{2\pi}d\theta\left(r^3\cos\theta+r^4\sin^2\theta\right)\Big|_1^2$$
$$=\int_0^{2\pi}d\theta(\cancel{7\cos\theta}+15\sin^2\theta)=\frac{15}{2}\cdot2\pi=15\pi \qquad \sin^2\theta=\frac{1-\cos2\theta}{2}$$
$A\supset D$

Let $D=\{(r,\theta)\in\mathbb{R}^2 \mid \alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta)\}$

Then
$$\boxed{\iint_D f=\int_\alpha^\beta d\theta\int_{h_1(\theta)}^{h_2(\theta)}f(r\cos\theta,r\sin\theta)\,r\,dr}$$

Area of D
$$\iint_D 1=\int_\alpha^\beta d\theta\int_{h_1(\theta)}^{h_2(\theta)}1\,r\,dr=\int_\alpha^\beta\frac12\left(h_2^2(\theta)-h_1^2(\theta)\right)d\theta$$

Ex $r(\theta)=\cos2\theta$
$$\text{Area}=\int_0^{\pi/2}\frac12\cos^22\theta\,d\theta=\frac14\int_0^{\pi/2}(1+\cos4\theta)d\theta=\frac{\pi}{8}$$

Improper integrals in polar coordinates

$$I=\int_{-\infty}^{+\infty}e^{-x^2}dx=\sqrt\pi$$
$$I^2=I\cdot I=\left(\int_{-\infty}^{+\infty}e^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty}e^{-y^2}dy\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{\overbrace{-x^2-y^2}^{-r^2}}\underbrace{dx\,dy}_{r\,dr\,d\theta}$$
$$=\int_0^{2\pi}d\theta\int_0^\infty dr\,e^{-r^2}r=\frac12\int_0^{2\pi}d\theta\int_0^\infty e^{-u}du=\frac12\int_0^{2\pi}d\theta\left[e^{-u}\right]_0^\infty=\pi$$

(15.4) Applications

Mass and density

$m=\rho\cdot V$ (density) $\quad m=\iint_A\rho(x,y)\,dx\,dy$

$Q=\iint_A\rho_c(x,y)\,dx\,dy$ (charge)

$\rho_c(x,y)=xy$, total charge; $y=1-x$
$$Q=\iint_D xy\,dxdy=\int_0^1dx\int_{1-x}^1xy\,dy=\int_0^1dx\left(\frac{xy^2}{2}\right)\Big|_{1-x}^1=\int_0^1\left(\frac x2-\frac{x(1-x)^2}{2}\right)dx$$
$$=\frac12\int_0^1(2x^2-x^3)dx=\frac12\left(\frac23x^3-\frac{x^4}{4}\right)\Big|_0^1=\frac12\left(\frac23-\frac14\right)=\frac{5}{24}$$

Moments and Center of Mass

center of mass
$$x_{cm}=\frac1m\iint_D x\,\rho(x,y)\,dxdy=\frac{\iint_D x\rho\,dA}{\iint_D\rho\,dA} \qquad \iint_D x\rho\,dA=M_y \text{ - moment about } y \text{ axis}$$
$$y_{cm}=\frac1m\iint_D y\,\rho(x,y)\,dxdy=\frac{\iint_D y\rho\,dA}{\iint_D\rho\,dA} \qquad \iint_D y\rho\,dA=M_x \text{ - moment about } x\text{-axis}$$

Ex $\rho(x,y)=1+3x+y$ $\quad y=2-2x$; $(0,2)$, $(0,0)$, $(1,0)$
$$x_{cm}=\frac{\iint x(1+3x+y)dxdy}{\iint(1+3x+y)dxdy}=\frac{M_y}{m}=\frac{1}{\frac83}=\frac38 \qquad y_{cm}=\frac{M_x}{m}=\dots=\frac{11}{16}$$
$$m=\int_0^1dx\int_0^{2-2x}(1+3x+y)dy=\int_0^1dx\left(y+3xy+\frac{y^2}{2}\right)\Big|_0^{2-2x}=\int_0^1\left(2-2x+6x-6x^2+\frac{4-8x+4x^2}{2}\right)=4\int_0^1(1-x^2)dx$$
$$=4\left(x-\frac{x^3}{3}\right)\Big|_0^1=4\left(1-\frac13\right)=\frac83$$
$$M_y=\int_0^1dx\int_0^{2-2x}(x+3x^2+xy)dy=\int_0^1dx(x-x^3)=\dots1$$

Moment of inertia

$\vec F=m\vec a$, $\vec F=\frac{d\vec p}{dt}$ $\quad \boxed{I=mR^2}$ $\quad \vec\tau=0$

$\vec r\times\vec F=I\vec\alpha$; $\quad rF\sin\alpha=mr^2\alpha$; $\quad F_{eff}=ma$

rotation $\alpha=\theta''$ $\quad \vec r\times\vec F$ - torque $\quad \vec\tau=I\vec\alpha$ - angular deceleration; moment of inertia $\vec L=I\vec\omega$, $\vec\tau=\frac{d\vec L}{dt}$

$\alpha=\frac{d^2\theta}{dt^2}$ $\quad r\alpha=r\frac{d^2\theta}{dt^2}=\frac{d^2s}{dt^2}=a$

$dI=dm\,r^2=\rho(x,y)\,dA\,r^2$ $\qquad \boxed{I=\iint_A dI=\iint_A r^2\rho\,dA}$

$\rho(x,y)=\rho$, $m=\rho\pi R^2$
$$I=\iint_D\rho\,r^2dA=\int_0^{2\pi}d\theta\int_0^R\rho r^3dr=2\pi\rho\frac{R^4}{4}=\pi\rho\frac{R^4}{2}=\frac12\pi\rho R^2R^2=\frac12mR^2$$

$I_x=\frac1{12}mL^2$ $\quad I\mapsto I_x+md^2$

Expectation value

$f(x,y)$ - probability distribution $\quad \sim e^{-\frac{x^2}{2\sigma}}$
$$\langle\varphi(x,y)\rangle=\frac{\int_V\varphi(x,y)f(x,y)dxdy}{\int_V f(x,y)dxdy} \qquad 1=\int_V f(x,y)dxdy$$

QM $\hat x$, $\hat p=-i\hbar\frac{\partial}{\partial x}$ $\quad \psi(\vec x)$ - amplitude of probability; $|\psi(x)|^2$ - probability
$$\langle\varphi(x,p)\rangle=\langle\psi|\hat\varphi|\psi\rangle=\frac{\iiint_{\mathbb{R}^3}\psi^*\hat\varphi\,\psi\,d^3x}{\iiint_{\mathbb{R}^3}|\psi(x)|^2d^3x} \qquad \iiint_{\mathbb{R}^3}|\psi(x)|^2d^3x=1 \text{ (unitarity)}$$

(15.5) Surface area

$z=f(x,y)$, $(x,y)\in D\subset\mathbb{R}^2$ $\quad \Delta x\Delta y=\Delta A$ $\quad \Delta x_i=x_{i+1}-x_i$, $\Delta y_i=y_{i+1}-y_i$

$\Delta T=|\vec a\times\vec b|$
$\vec a_i=(\Delta x_i,\ 0,\ f_x(x_i,y_i)\Delta x_i)$
$\vec b_i=(0,\ \Delta y_i,\ f_y(x_i,y_i)\Delta y_i)$
$$\vec a_i\times\vec b_i=\begin{vmatrix}\vec i&\vec j&\vec k\\ \Delta x_i&0&f_x\Delta x\\ 0&\Delta y_i&f_y\Delta y\end{vmatrix}=(-f_x\Delta y_i\Delta x_i,\ -f_y\Delta x_i\Delta y_i,\ \Delta x_i\Delta y_i)=\Delta A_i(-f_x,-f_y,1)$$
$$|\vec a_i\times\vec b_i|=\sqrt{1+f_x^2+f_y^2}\,\Delta A_i$$

Calculating Riemann sum $\displaystyle\sum_i\sum_j|\vec a_i\times\vec b_i|\xrightarrow[\Delta y\to0]{\Delta x\to0}\boxed{A=\iint_D\sqrt{1+f_x^2+f_y^2}\,dxdy}$

Ex Area $z=x^2+y^2$, $0\le z\le10$ $\quad z'_x=2x$, $z'_y=2y$
$$A=\iint_D\sqrt{1+4x^2+4y^2}\,dxdy=\int_0^{2\pi}d\theta\int_0^{\sqrt{10}}\sqrt{1+4r^2}\,r\,dr=2\pi\frac18\int_1^{41}u^{1/2}du=\frac\pi4\frac{u^{3/2}}{3/2}\Big|_1^{41}=\frac\pi6\left(41^{3/2}-1\right)$$
$u=1+4r^2$, $du=8r\,dr$; $r=0$, $u=1$; $r=\sqrt{10}$, $u=41$

(15.5) Triple integrals (n-integrals)

partitions $P_x\times P_y\times P_z=P$ $\quad x_i^*,y_j^*,z_k^*$ $\quad i,j,k\in P$

$\Delta V_{ijk}=\Delta x_i\,\Delta y_j\,\Delta z_k$
$$S=\sum_i\sum_j\sum_k f(x_i^*,y_j^*,z_k^*)\,\Delta V_{ijk}$$
$f(x,y,z)$, $(x,y,z)\in B=[a,b]\times[c,d]\times[e,f]$, continuous
$$\boxed{\iiint_B f(x,y,z)\,dV=\lim_{\text{mesh}(P_x,P_y,P_z)\to0}S}$$

In fact, take $B^n=[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_n,b_n]$ with partitions $P_1,P_2,\dots,P_n$
$$S=\sum_{i_1,\dots,i_n}f(x_{i_1}^*,\dots,x_{i_n}^*)\,\Delta x_{i_1}\cdots\Delta x_{i_n}\xrightarrow[\forall i]{\text{mesh }P_i\to0}\iint\cdots\int_{B^n}f(x_1,\dots,x_n)\,d^nx=\int_{B^n}f(x_1,\dots,x_n)\,d^nx$$
$d^nx=dx_1\cdots dx_n$

Fubini theorem for triple integrals
$$\int_B f(x,y,z)\,dV=\int_a^b dx\int_c^d dy\int_e^f dz\,f(x,y,z)$$

Ex $S=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$ $\quad z=u_2(x,y)$, $z=u_1(x,y)$, $dA=dxdy$

$f(x,y,z)$ $\quad S\to\mathbb{R}$
$$\boxed{\int_S f\,dV=\iint_D dA\int_{u_1(x,y)}^{u_2(x,y)}dz\,f(x,y,z)}$$

$S=\{(x,y,z)\in\mathbb{R}^3 \mid c\le y\le d,\ h_1(y)\le x\le h_2(y),\ u_1(x,y)\le z\le u_2(x,y)\}$ $\quad y=c$, $y=d$, $x=h_1(y)$, $x=h_2(y)$
$$\boxed{\int_S f\,dV=\int_c^d dy\int_{h_1(y)}^{h_2(y)}dx\int_{u_1(x,y)}^{u_2(x,y)}dz\,f(x,y,z)}$$

Ex T: $x+y+z=1$, $z=1-x-y$; $x=0$, $y=0$, $z=0$; $(0,0,1)$, $(0,1,0)$, $(1,0,0)$
$$I=\int_T z\,dV$$
$$I=\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}z\,dz=\frac12\int_0^1dx\int_0^{1-x}dy\,(1-x-y)^2=\frac12\int_0^1dx\int_{1-x}^0(-du)\,u^2=\frac12\int_0^1dx\int_0^{1-x}u^2du$$
$$=\frac16\int_0^1dx\,(1-x)^3=\frac16\left(-\frac14\right)(1-x)^4\Big|_0^1=\frac1{24}$$
$u=1-x-y$, $dy=-du$; $y=0$: $u=1-x$; $y=1-x$: $u=0$

$T=\{\vec x\in\mathbb{R}^3 \mid (x,y)\in D,\ 0\le z\le1-x-y\}$

$D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le1,\ 0\le y\le1-x\}$ $\quad x+y=1$

(15.7) (15.8) Cylindrical coordinates and Spherical

Cartesian $\to$ Cylindrical: $x=r\cos\theta$, $y=r\sin\theta$, $z=z$

$$\iiint f(x,y,z)\,dxdydz\ \to\ \iiint f(r\cos\theta,r\sin\theta,z)\cdot r\,dr\,d\theta\,dz$$

Ex S bounded by Cylinder $x^2+y^2=1$, plane $z=4$, paraboloid $z=1-x^2-y^2$; $\rho=$ const
$$M=\int_S\rho\,dV=\rho\int_0^{2\pi}d\theta\int_0^1dr\int_{1-r^2}^4dz\cdot r=\rho\,2\pi\int_0^1r\,dr\int_{1-r^2}^4dz=2\pi\rho\int_0^1r\,dr(4-1+r^2)=2\pi\rho\int_0^1(3r+r^3)dr$$
$$=2\pi\rho\left(\frac32r^2+\frac14r^4\right)\Big|_0^1=2\pi\rho\left(\frac32+\frac14\right)=\frac72\pi\rho$$

Change of variables in integrals

Differential forms $S\subset\mathbb{R}^2$, $(x,y)\in S$
- $F(x,y)$ - 0-form
- $f(x,y)\,dx+g(x,y)\,dy$ - 1-form on S
- $dx\wedge dy$ - 2-form

$\wedge$ - exterior product $\quad dx\wedge dy=-dy\wedge dx$, $\quad dx\wedge dx=-dx\wedge dx=0$

$\Omega^d(M)$ - space of differential d-forms on M

$x\to-x$ $\quad$ Orientation in $\mathbb{R}^n$: order of $x_1,x_2,x_3,\dots$

3-form $f(x,y,z)\,dx\wedge dy\wedge dz$

De Rham complex (de Rham differential)
$$0\to\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\Omega^2(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^n(M)\xrightarrow{d}0$$
$F\mapsto dF\mapsto d(dF)$

$$d(f(x,y)dx+g(x,y)dy)=df\wedge dx+dg\wedge dy=(f_x\cancel{dx}+f_ydy)\wedge dx+(g_xdx+g_y\cancel{dy})\wedge dy=f_y\,dy\wedge dx+g_x\,dx\wedge dy=(g_x-f_y)\,dx\wedge dy$$

$d^2=0$ $\quad \boxed{d^2F=0}$ (differentiable on S)
$$d(dF(x,y))=d(\underbrace{F_x}_{f}dx+\underbrace{F_y}_{g}dy)=F_{xx}\cancel{dx\wedge dx}+F_{xy}\,dy\wedge dx+F_{yx}\,dx\wedge dy+F_{yy}\cancel{dy\wedge dy}=(F_{xy}-F_{yx})\,dy\wedge dx=0$$

volume form $\iint_S f(x,y)\,dx\,dy$ $\quad x=u(x,y)$, $y=v(x,y)$

$dx=u_x\,dx+u_y\,dy$, $\quad dy=v_x\,dx+v_y\,dy$

$dx\wedge dy\to(\#)\,du\wedge dv$

$dx\wedge dy=u_xv_y\,dx\wedge dy+u_yv_x\,dy\wedge dx=(u_xv_y-u_yv_x)\,dx\wedge dy$ $\quad \begin{vmatrix}u_x&u_y\\v_x&v_y\end{vmatrix}$ - Jacobian

Jacobi matrix $\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}$
$$\frac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1}&\cdots&\frac{\partial u_1}{\partial x_n}\\ \vdots&&\vdots\\ \frac{\partial u_n}{\partial x_1}&\cdots&\frac{\partial u_n}{\partial x_n}\end{vmatrix}$$

Ex polar coordinates $x=x(r,\theta)=r\cos\theta$, $y=y(r,\theta)=r\sin\theta$
$$J=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta&-r\sin\theta\\ \sin\theta&r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r$$
$dx\wedge dy\to r\,dr\wedge d\theta$

Ex $x_1=x_1(u_1,\dots,u_n)$, $\dots$, $x_n=x_n(u_1,\dots,u_n)$
$$dx_1\wedge dx_2\wedge\dots\wedge dx_n=\sum_{i_1,\dots,i_n}(\ )\wedge(\ )\wedge\dots\wedge(\ )\,(-1)^{\sigma_{i_1\dots i_n}}=J\,du_1\wedge\dots\wedge du_n$$

Problem 1 $(x,y,z)\to(r,\theta,\phi)$: $x=ar\sin\phi\cos\theta$, $y=br\sin\phi\sin\theta$, $z=cr\cos\phi$ $\quad J=abc\,r^2\sin\phi$

Problem 2 Volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le1$ in a smart way
$$Vol=\iiint1\,dxdydz=abc\int_0^{2\pi}d\theta\int_0^\pi d\phi\int_0^1r^2\sin\phi=\frac{4\pi}{3}abc$$

Problem 3 $\iint_Q\frac1{x+y}\,dA$ $\quad Q$: $(1,0),(4,0),(0,1),(0,4)$ $\quad x=u-uv$, $y=uv$

Problem 1
$$\begin{vmatrix}a\sin\phi\cos\theta&-r\sin\phi\sin\theta&r\cos\phi\cos\theta\\ a\sin\phi\sin\theta&r\sin\phi\cos\theta&r\cos\phi\sin\theta\\ a\cos\phi&0&-r\sin\phi\end{vmatrix}=\cos\phi\cdot r^2\begin{vmatrix}-\sin\phi\sin\theta&\cos\phi\cos\theta\\ \sin\phi\cos\theta&\cos\phi\sin\theta\end{vmatrix}-r^2\sin\phi\begin{vmatrix}\sin\phi\cos\theta&-\sin\phi\sin\theta\\ \sin\phi\sin\theta&\sin\phi\cos\theta\end{vmatrix}$$
$$r^2(-\sin\phi\cos^2\phi)\qquad -r^2\sin^3\phi \qquad =-r^2\sin\phi$$

$dx\wedge dy\wedge dz=-r^2\sin\phi\,dr\wedge d\theta\wedge d\phi$

$$\boxed{\int f\,dx_1\cdots dx_n=\int f\left|\frac{\partial(x_1,\dots,x_n)}{\partial(u_1,\dots,u_n)}\right|du_1\cdots du_n}$$

2d $x=x(u,v)$, $y=y(u,v)$ $\quad du\wedge dv$, $dx\wedge dy$ $\quad |\vec r_u\times\vec r_v|$

$x=u-uv$, $y=uv$; $x=0$, $y=\alpha x$, $x+y=4$, $x+y=1$, $y=0$; $x=x(u,v)$, $y=y(u,v)$
$$\int_Q\frac{dxdy}{x+y}=\int_Q\frac1u\begin{vmatrix}1-v&-u\\ v&u\end{vmatrix}dudv=\int_P\frac{|u-uv+uv|}{u}du\,dv=\int_P du\,dv=\left(\int_1^4du\right)\left(\int_0^1dv\right)=3$$

$x+y=u$
$y=0\Rightarrow uv=0\Rightarrow v=0$
$x=0\Rightarrow u(1-v)=0\Rightarrow v=1$
$y=\alpha x\Rightarrow \cancel{u}v=\alpha\cancel{u}(1-v)$, $v=\alpha-\alpha v$, $(1+\alpha)v=\alpha$, $\boxed{v=\frac{\alpha}{1+\alpha}}$

$1\le u\le4$