On causality and NEC in Lifshitz holography

Peter Koroteev

University of Minnesota

In collaboration with M. Libanov, C. Hoyos

0712.1136, 0901.4347, 1007.1428

GLS 2011 Univ. of Chicago April 30th

Outline

- Lifshitz solutions and bulk spectrum at different critical exponents z
- Null Energy Condition (NEC) and Causality
- Constraints on z
- NEC and higher derivative Gravity

Theories with dynamical scaling

$$\mathcal{L} = (\partial_t \phi)^2 - c^2 \ell^{2(z-1)} \phi (-\partial_{\mathbf{x}}^2)^z \phi \qquad t \to \lambda^z t, \;\; x \to \lambda x$$

Dispersion relation

$$\omega^2 = \frac{c^2}{\ell^2} (\ell k)^{2z}$$

Phase velocity

$$v_{\rm ph} = \frac{\omega}{k} = c(\ell k)^{z-1}$$

Physical dimensions

$$[\phi] = \frac{d-1}{2}, \; [\omega] = 1, \; [k] = 1, \; [\ell] = 2(z-1)$$

Scaling dimensions

$$[[\phi]] = \frac{d-z}{2}, \; [[\omega]] = z, \; [[k]] = 1, \; [[\ell]] = 0$$

Lifshitz metric

$$ds^2 = \frac{L^2}{r^2} \left(-\frac{\kappa^2 dt^2}{r^{2(z-1)}} + dr^2 + d\mathbf{x}^2 \right)$$

Speed of light

$$c(r) = \frac{\kappa}{r^{z-1}}$$

Same dependence of r

$$c(r) = c\, \ell^{(z-1)} r^{-(z-1)}$$

what is the difference between z>1 and z<1 from the gravitational perspective?

Lifshitz solution

Take perfect fluid in the bulk

$$T^\mu_\nu = (p+\rho)u^\mu u^\nu - p\delta^\mu_\nu$$

equation of state

$$p = w\rho$$

introduce anisotropy

$$T^0_0 = (1+\omega)\rho u_0 u^0 - p_{d+1}$$
$$T^1_1 = (1+w)\rho u_1 u^1 - p_1$$
$$T^{d+1}_{d+1} = (1+\omega)\rho u_{d+1} u^{d+1} - p_{d+1}$$
$$T^0_{d+1} = (1+\omega)\rho u^0 u_{d+1}$$

equations of state

$$p = \omega\rho$$
$$p = w\rho$$
$$p = \omega\rho$$
$$p = \omega\rho$$

[PK, Libanov]

d=3

$$S = -\int \frac{1}{e^2} F_{(2)} \wedge *F_{(2)} + F_{(3)} \wedge *F_{(3)} - c\int F_{(2)} \wedge B_{(2)}$$

$$F_{(2)} = dA_{(1)},\ F_{(3)} = dB_{(2)}$$

$$F_{(2)} = A\,\theta_r \wedge \theta_t\,, \qquad F_{(3)} = B\,\theta_r \wedge \theta_x \wedge \theta_y$$

[Kachru, Liu, Mulligan]

give the Lifshitz solution

$$ds^2 = L^2\left(-\frac{dt^2}{r^{2\xi}} + \frac{d\mathbf{x}^2}{r^{2\zeta}} + \frac{dr^2}{r^2}\right)$$

$$\rho = -\Lambda - \frac{d(d-1)}{2L^2}\zeta^2$$
$$w = -1 + \frac{(\xi + (d-2)\zeta)(\xi-\zeta)}{L^2\rho}$$
$$\omega = -1 + \frac{(d-2)\zeta(\xi-\zeta)}{L^2\rho}$$

$$\Lambda = -\frac{z^2+z+4}{2L^2}$$
$$A^2 = \frac{2z(z-1)}{L^2}$$
$$B^2 = \frac{4(z-1)}{L^2}$$

Spectrum at different z

states created by a scalar operator $\langle \mathcal{O}_\varphi \rangle \sim \phi(r)$ classical normalizable solutions

Metric

$$ds^2 = du^2 + e^{2A(u)}(-e^{2B(u)}dt^2 + d\mathbf{x}^2)$$

Bulk scalar action

$$S = -\int d^{d+2}x\sqrt{-g}\left(\partial_M\Phi\,\partial^M\Phi + m^2\Phi^2\right)$$

Equation of motion

$$\phi'' + ((d+1)A' + B')\phi' + e^{-2A-2B}\omega^2\phi - e^{-2A}k^2\phi - m^2\phi = 0$$

after some redefinitions reduces to Schoedinger equation

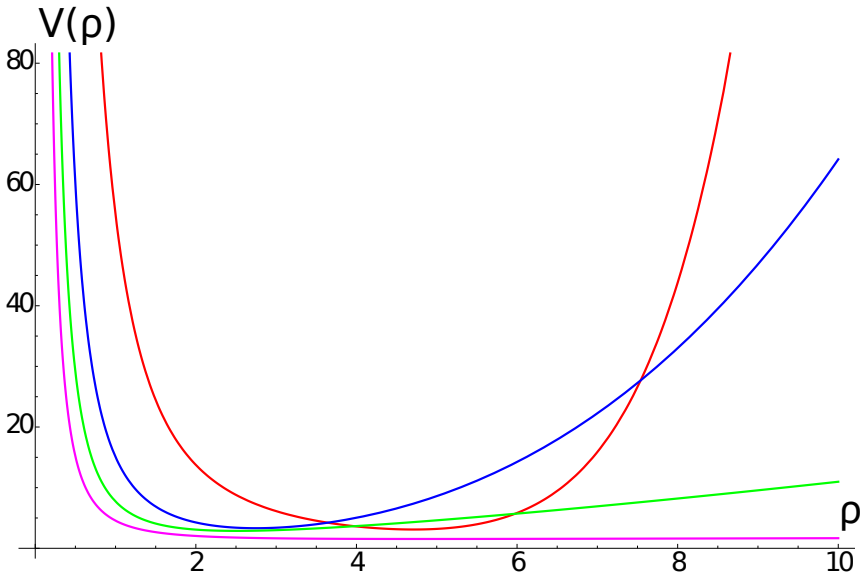
$$-\ddot{\psi} + V(\rho)\psi = \omega^2\psi$$

in the potential

$$V(\rho) = \frac{d^2 + 2dz + 4m^2L^2}{4\rho^2 z^2} + k^2\left(\frac{\rho z}{L}\right)^{\frac{2}{z}-2}$$

z>1 continuous

z<1 discrete



WKB analysis

Find the value of the turning point in the limit $\omega \to \infty$

The condition $V(\rho_0) = \omega^2$ leads to

$$v_{wf} \simeq v_{ph} = \frac{\omega}{k} \simeq e^{B(\rho_0)}$$

Thus the wavefront velocity is given by the local speed of light at the turning point.

Therefore plane wave states created by a scalar operator in the field theory have wavefront velocities that are equal to the local speed of light in the holographic dual.

Growing vs. decreasing s.o.l.

z<1

$$ds^2 = \frac{L^2}{r^2}\left(dr^2 + d\mathbf{x}^2 - r^{2-2z}\kappa^2 dt^2\right)$$

Boundary is d-dimensional
Conical singularity for z=1/2

null geodesics tangent to boundary

$$\frac{dt}{dr} = -\frac{r^{z-1}}{\kappa}\,, \quad t(r_0) = 0$$

z>1

$$ds^2 = \frac{L^2}{z^2 R^2}\left(-\kappa^2 dt^2 + dR^2 + R^{2-2/z} d\mathbf{x}^2\right)$$

Boundary goes along time direction
Conical singularity for z=2

null geodesics orthogonal to boundary

$$t(r) = \frac{r_0^z - r^z}{z\kappa}$$

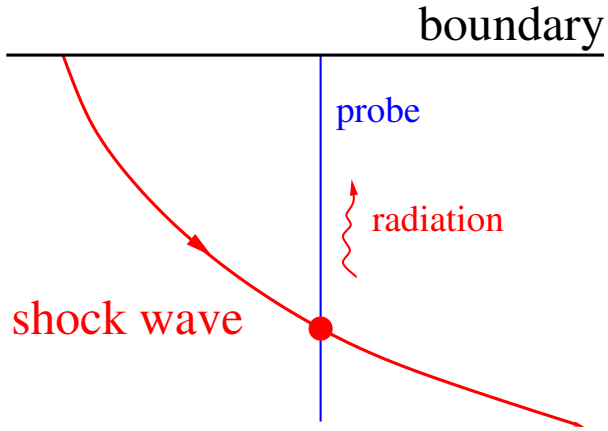
boundary singularity suggests UV completion

For our purpose it will be enough to introduce a cutoff, since the results we will obtain are independent on how the ultraviolet theory is defined

Causality from shock waves

[Hoffman,Maldacena]

source in the field theory localized in time
and in one of the spatial directions



Null geodesics

$$\frac{dt}{dr} = \frac{E r^{2(z-1)}}{\kappa^2 \sqrt{\frac{E^2 r^{2(z-1)}}{\kappa^2} - P^2}}\,, \qquad \frac{dx}{dr} = \frac{P}{\sqrt{\frac{E^2 r^{2(z-1)}}{\kappa^2} - P^2}}$$

for z>1

$$t \simeq \frac{r^z}{z\kappa} \to \infty\,, \quad x \simeq \frac{\kappa P}{(2-z)E} r^{2-z} + x_0$$

z<1

$$\left(\frac{r_0}{\ell}\right)^{1-z} = \frac{E}{cP}$$

right turning point

The shock wave will be a source of radiation of gravitational fields that will then propagate along the radial direction to the boundary, producing a front of radiation that can be interpreted as the front of the perturbation in the dual theory

Calculate the time and position of the shockwave travelled back to the boundary

for z>1

the shock wave travels faster than light signals at the boundary

$$v_s > c$$

Null Energy Condition

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$$

Example - perfect fluid $p = w\rho$ NEC $w > -1$

cosmological constant $w = 1$

Broken NEC is usually associated with *superluminal propagation, causality violation, etc*

From Einstein equations NEC $R^t_t - R^x_x \leq 0\,, \quad R^t_t - R^u_u \leq 0$

$$ds^2 = du^2 + e^{2A(u)}(-e^{2B(u)}dt^2 + d\mathbf{x}^2)$$

Ricci tensor

$$R^t_t = -B'' - DA'B' - B'^2 - A'' - (D-1)A'^2$$
$$R^x_x = -A'B' - A'' - (D-1)A'^2$$
$$R^u_u = -B'' - (A'+B')^2 - (D-1)A'' - (D-2)A'^2$$

NEC

$$B'' + B'(B' + (D-1)A') \geq 0$$

For Lifshitz Bulk NEC $\mathbf{z \geq 1}$

NEC and speed of light

let's check our holographic construction: 1 Bulk NEC; 2 Boundary NEC

$$ds^2 = du^2 + e^{2A(u)}(-e^{2B(u)}dt^2 + d\mathbf{x}^2)$$

NEC 1 $$B'' + B'(B' + (D-1)A') \geq 0$$

define $$B' = Ce^{-(D-1)A-B}$$

The derivative of the local speed of light is

$$(e^B)' = B'e^B = Ce^{-(D-1)A}$$

C>0 speed of light is monotonically increasing

C<0 speed of light is monotonically decreasing

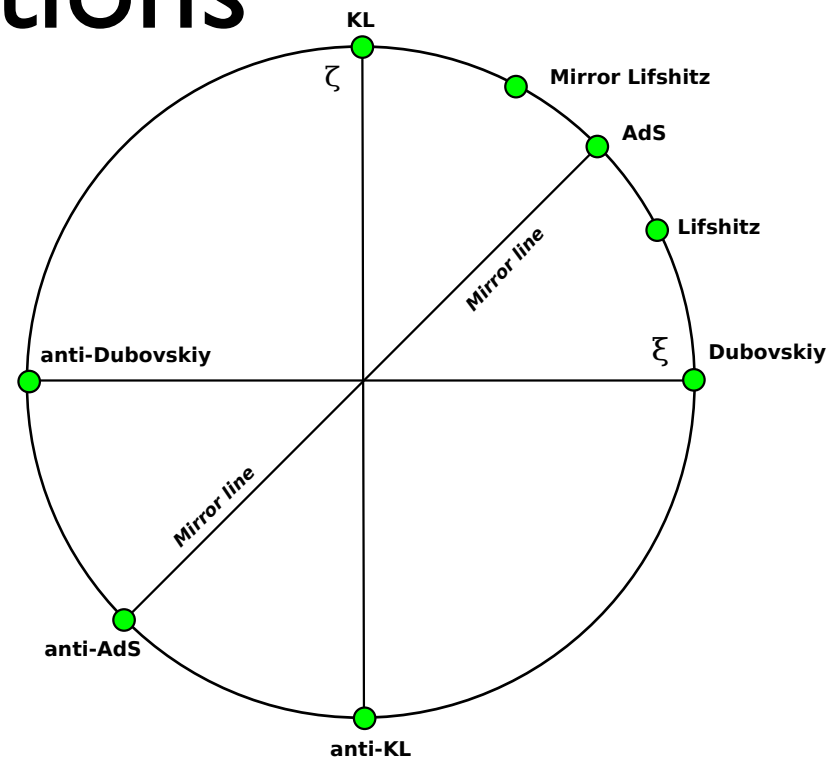
For Lifshitz $\mathbf{z \geq 1}$ implies both bulk and boundary NEC

Generically NEC is necessary in order to have a consistent holographic description

2-point functions

Scalar EOM

$$\varphi'' - \frac{z+d-1}{r}\varphi' + \frac{\omega^2}{\kappa^2} r^{2(z-1)}\varphi - k^2\varphi - \frac{m^2}{r^2}\varphi = 0$$



Correlator

$$G_2(\omega, \mathbf{k}) = -\lim_{\ell\to 0} \sqrt{-g} g^{rr} \varphi'_{\omega,\mathbf{k}}(r)\varphi_{\omega,\mathbf{k}}(r)\big|_{r=\ell}$$

Scaling dimension $\quad m^2 L^2 = \Delta(\Delta - d - z)$

z=2

$$G_2(\omega, k) \simeq \left(\frac{4\omega^2}{\kappa^2} + k^4\right)\left[\log(i\kappa\omega) + \psi\left(\frac{3}{2} - \frac{i\kappa k^2}{4\omega}\right) + i\Theta(\mathrm{Im}\,\omega)\pi\,\mathrm{sech}\left(\frac{\kappa k^2 \pi}{4\omega}\right)\right]$$

branch cut along the positive imaginary axis $\quad \omega_n = \dfrac{i\kappa k^2}{4n+6}, \quad n = 0, 1, 2, \ldots$

z=1/2

$$G_2(\omega, k) \simeq k^{5/2} \frac{\Gamma\left(\frac{7}{4} - \frac{\omega^2}{2k\kappa^2}\right)}{\Gamma\left(-\frac{3}{4} - \frac{\omega^2}{2k\kappa^2}\right)} \qquad \omega_n^2 = \left(2n + \frac{7}{2}\right)\kappa^2 k, \quad n = 0, 1, 2, \ldots$$

phase velocity $\quad (v_{ph})_n = \dfrac{\omega}{k_n} = \dfrac{c}{\omega\ell}\left(2n + \dfrac{7}{2}\right)$ $\quad$ leads to superluminal propagation

Further Constraints

Equation of state in scale invariant theory

$$z\langle T_{tt}\rangle - d\langle T_{xx}\rangle = 0$$

we get more constraints

$$\mathbf{1 \leq z \leq d}$$

Bulk (holographic) NEC

Boundary DEC

Assuming dominant energy condition (DEC)

$$-T^0_\nu \xi^\nu > 0 \quad \langle T_{tt}\rangle \geq 0$$

For the boundary theory which respects DEC

$$\langle T_{tt}\rangle - |\langle T_{xx}\rangle| = \langle T_{tt}\rangle \left(1 - \frac{z}{d}\right) \geq 0$$

Nevertheless, real condensed matter systems with z>d are known

[Si et al 2001]

Also there are top-bottom constructions in 2+1 with z~ 39 [Gauntlet et al 2010]

NEC and Higher Derivative Gravity

$$S = \int d^D x \sqrt{g} \left(R - 2\Lambda + L^2 \beta_1 R^2 + L^2 \beta_2 R_{\alpha\beta} R^{\alpha\beta} + L^2 \beta_3 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right)$$

Represent higher derivative stuff as 'source'

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = L^2 \Theta_{\mu\nu}$$

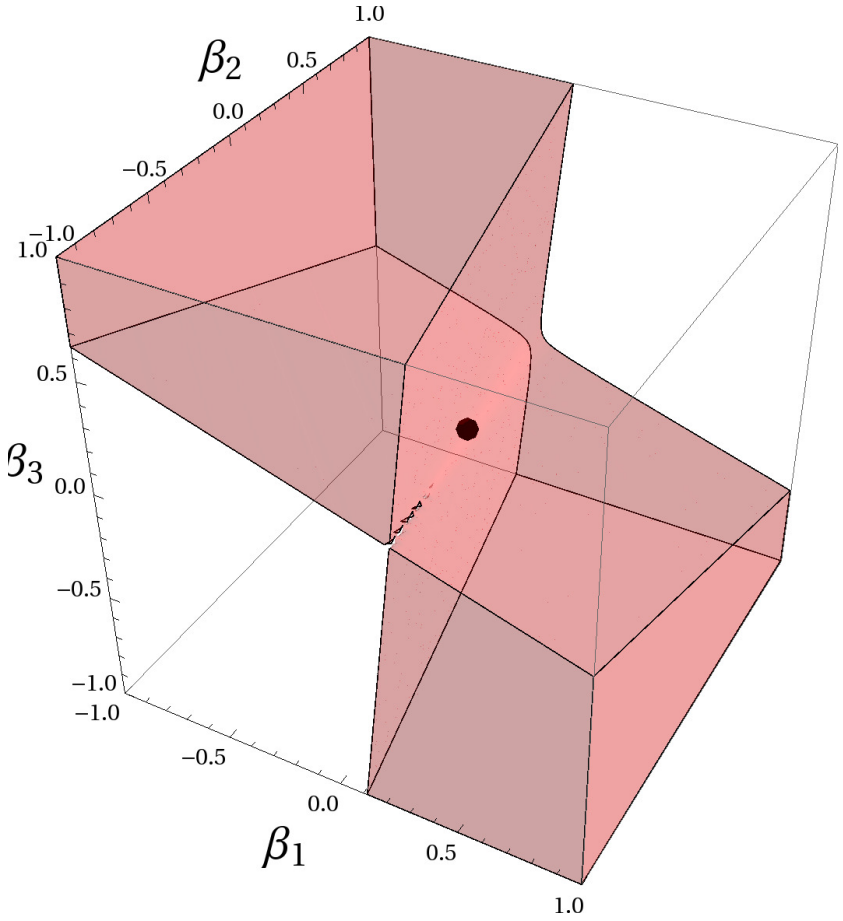
Constraints on existence of Lifshitz solutions

$$\Lambda = -\frac{1}{L^2} \left[1 + 2(\beta_1 - \beta_3) + 2z + \left(1 - 2z + \frac{1}{2} z^4 \right) (4\beta_1 + 2\beta_2 + 4\beta_3) + (3z^2 - 2z^3)(\beta_2 + 4\beta_3) \right]$$

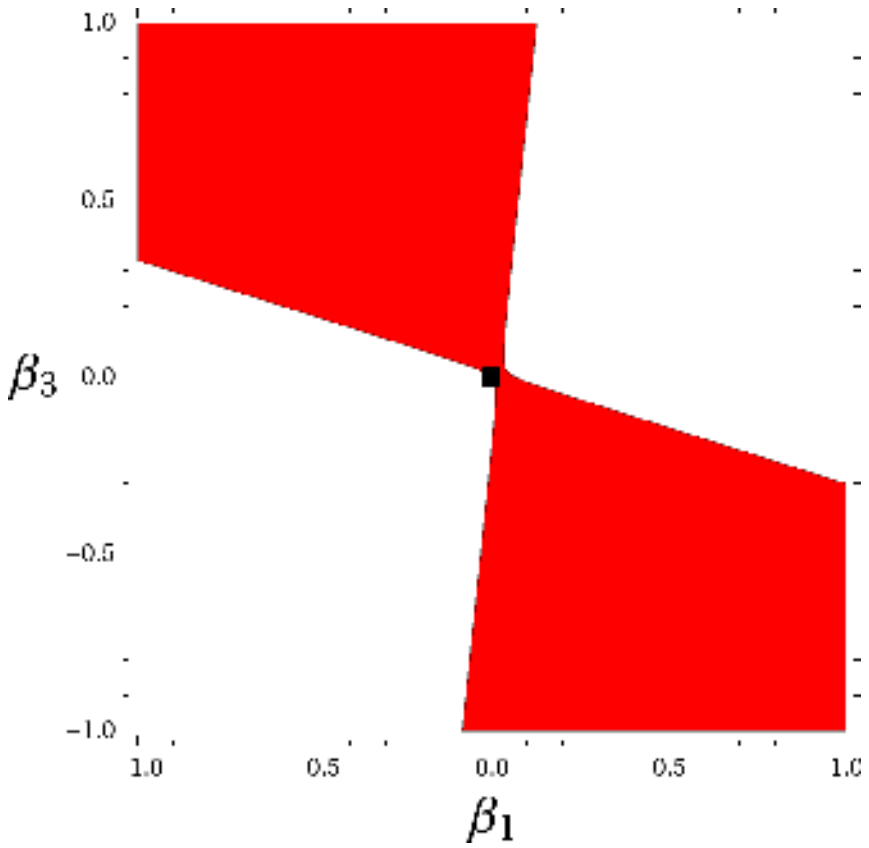
$$2(2z^2 + (D-2)(2z + D - 1))\beta_1 + 2(z^2 + D - 2)\beta_2 + 4(z^2 - (D-2)z + 1)\beta_3 = 1$$

Impose NEC on the rhs of the Einstein equations
treating is as a 'source' to Einstein Gravity

Solutions with z < 1 exist in the full region with fixed cosmological constant



$\beta_2 = 0$



violations of the NEC are possible in the full region !!

Conclusions

- Geometries produced by matter that violates the NEC will produce superluminal propagation in the dual theory on the boundary
- Inclusion if higher derivative corrections brings possible NEC violations
- Further role of NEC in holography and RG dynamics of field theories (modifications of a-theorem?)