Math 53 Multivariable Calculus

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Book] Stewart Calculus: Early transcendentals (1, Berkeley edition
2, 8th edition)

HW assigned every week (not graded)

Quizzes: Weeks 2,3,4,6,7 } 20%
Midterm: Week 5 } 30%
Final: Week 8 } 50%
Date

In this course

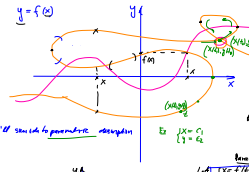
1. Parametric curves in 2D, Areas, lengths
2. 3D Coordinates, parametric curves in 3D
3. Vector Calculus: Derivatives, Integrals of Vector functions
4. Lagrange multipliers, Gradient ascent, ...
5. Stokes theorem, etc

Today Curves defined by parametric equations in 2D

What is a dimension?

0D: pt. 1D: 2D: 3D:

Previously: $y = f(x)$



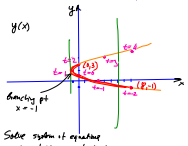
Simple-valued functions
multi-valued functions (complex analysis)

$y = x^2$

$x = \pm\sqrt{y}$

We'll switch to parametric description

Ex $\begin{cases} x = C_1 \\ y = C_2 \end{cases}$



Parametric equations

Let $\boxed{\begin{cases} x = f(t) \\ y = g(t) \end{cases}}$ t - parameter

$a \le t \le b$

a, b can be $\pm\infty$

$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$ $t \in \mathbb{R}$, $-\infty \le t \le \infty$

t	x	y
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

$-2 \le t \le 2$

$(x(t_1), y(t_1))$ - initial point

$(x(t_2), y(t_2))$ - terminal pt

Solve system of equations

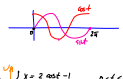
$y = t + 1 \Rightarrow t = y - 1$

$x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2$

$\boxed{x = y^2 - 4y + 3}$

Ex $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $0 \le t \le 2\pi$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$



Ex $\begin{cases} x = \cos\frac{3t}{2} \\ y = \sin\frac{3t}{2} \end{cases}$ $0 \le t \le 2\pi$

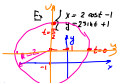
$t = \pi$

$u = 3t$ $0 \le u \le 3\pi$

$\begin{cases} x = \cos u \\ y = \sin u \end{cases}$ $= 2\pi/3$

Ex $\begin{cases} x = 2\cos t - 1 \\ y = 2\sin t + 1 \end{cases}$ $0 \le t \le 2\pi$

$\begin{cases} \frac{x+1}{2} = \cos t \\ \frac{y-1}{2} = \sin t \end{cases}$



t	x	y
0	1	1
π/2	-1	3
π	-3	1
3π/2	-1	-1
2π	1	1

$u = 0 \Rightarrow x = 1$

$v = 0 \Rightarrow y = 1$

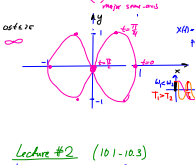
Ex $\begin{cases} x = 3\cos t - 5 \\ y = \sin t + 3 \end{cases}$ $0 \le t \le 2\pi$

ellipse

$\frac{x^2}{3^2} + y^2 = 1$

In general, an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ex $\begin{cases} x = \cos\frac{t}{2} \\ y = \sin\frac{3t}{2} \end{cases}$ $0 \le t \le 2\pi$



∞

$x(t) = \cos\left(\frac{3}{2}t\right)$ frequency

period $T = \frac{4\pi}{3}$

t	x	y
0	1	0
π/2	1/√2	1
π	0	0
3π/2	-1/√2	-1
π	1	0

$t_0 = \pi$

$T = T_2$

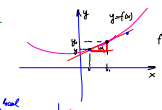
$\cos(t_0 + T) = \cos(t_0)$

Fun problem

Calculate $\int \frac{dx}{dx} = \frac{1}{a}\log|x| + C$

Lecture #2 (10.1-10.3)

Tangents



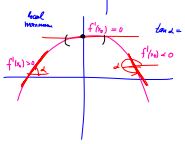
$x \to x_0$

$f(x) \to f(x_0)$

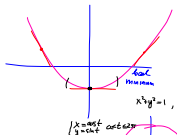
Slope $\frac{f(x)-f(x_0)}{x - x_0} \xrightarrow{x \to x_0} f'(x_0)$

$\tan\alpha\big|_{x=x_0} = f'(x_0)$

$y = f'(x_0)(x - x_0) + f(x_0)$



$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$



$x^2 + y^2 = 1$, $y = +\sqrt{1-x^2}$

Parametric equations and tangents

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $a \le t \le b$

$x(t), y(t)$ are differentiable on $[a, b]$

Before $y = f(x)$ tangent line passing through (x_0, y_0) on $y = f(x)$

$y = f'(x_0)(x - x_0) + y_0$

$\begin{cases} x = x(t_0) \\ y = y(t_0) \end{cases}$

$f'(x_0) = \frac{dy}{dx}\Big|_{x=x_0} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t_0)}{x'(t_0)}\Big|_{t=t_0}$

provided that $\frac{dx}{dt} \ne 0$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt} = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)} = \frac{\frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^2}}{x'(t)} = \frac{y''x' - y'x''}{(x')^3}$

Ex $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$ $-\infty < t < \infty$

$x = 0 \Rightarrow t = 0$

$y = 0 \Rightarrow t(t^2 - 3) = 0 \Rightarrow t = 0$, $t = \pm\sqrt{3}$

$t = \pm\sqrt{3}$, $(y = 0)$ $x = 3$

$x' = 2t$

$y' = 3t^2 - 3$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 3}{2t} = \boxed{\frac{3}{2}\left(t - \frac{1}{t}\right)}$

Horizontal tangent $\frac{dy}{dx} = 0$ $t - \frac{1}{t} = 0 \Rightarrow t^2 = 1$, $t = \pm 1$

Vertical tangent $x' = 0$, $2t = 0 \Rightarrow t = 0$

$t = \sqrt{3}$: $\frac{dy}{dx} = \frac{3}{2}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) > 0$

$t = -\sqrt{3}$ $\frac{dy}{dx} = \frac{3}{2}\left(-\sqrt{3} + \frac{1}{\sqrt{3}}\right) < 0$

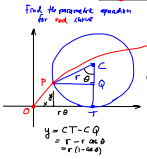
$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{3}{2}\left(t - \frac{1}{t}\right)\right)'}{2t} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3}{4}\frac{t^2+1}{t^3}$

$\frac{dy}{dx} > 0$ $t > 0$

$\frac{dy}{dx} < 0$ $t < 0$

Cycloid curve

Find the parametric equation for red curve



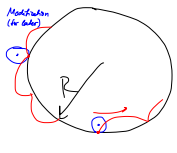
$\boxed{\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}}$ $0 \le \theta \le 2\pi$

OT = length of the arc PT $= r \cdot \theta$

$x = OT - PQ$
$= r\theta - r\sin\theta$
$= r(\theta - \sin\theta)$

$y = CT - CQ$
$= r - r\cos\theta$
$= r(1 - \cos\theta)$

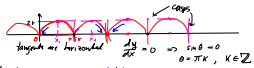
Hypotrochoid (the outer)



$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$

θ - parameter

tangent lines $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$



tangent are horizontal $\frac{dy}{dx} = 0 \Rightarrow \sin\theta = 0$

$\theta = \pi K$, $K \in \mathbb{Z}$

$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

Vertical lines $\frac{dx}{d\theta} = 0$ $1 - \cos\theta = 0$

$\begin{cases} x_0 = r\pi K \\ y_0 = 2r \end{cases}$

$\lim_{\theta \to 0} \frac{dy}{dx} = \lim_{\theta \to 0} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta \to 0} \frac{\cos\theta}{\sin\theta} = \infty$

$\frac{0}{0}$ $\frac{0}{0}$ L'Hospital

tangent lines are vertical at $\theta = 2\pi\ell$ $\ell \in \mathbb{Z}$

$\theta = \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

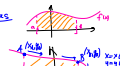
Lecture #3 (10.2-10.4)

Fun problem Find the pattern:

0123 → 1
0231 → 2
1323 → 3
2313 → 3
1120 → 1
1230 → 5
1137 → 0
0011 → 6

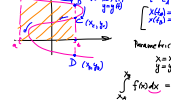
Finite cyclic group

Areas



Area $= \int_a^b f(x)dx = F(b) - F(a)$, $F'(x) = f(x)$

$x = x(t)$



$t = t_a$

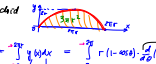
$\begin{cases} x(t_a) = x_a \\ y(t_a) = y_a \end{cases}$ $\begin{cases} y = y(x) \\ y = y(t) \end{cases}$

Parametric formula for the area

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t_a \le t \le t_b$ $dx = x'(t)dt$

$\int_{x_a}^{x_b} f(x)dx = \int_{t_a}^{t_b} f(x(t))x'(t)dt$

Ex Cycloid



$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $0 \le \theta \le 2\pi$

$x(t) = 0 \Rightarrow t = 0$

$x(t) = 2\pi r \Rightarrow t = 2\pi$

$dx = r(1 - \cos\theta)d\theta$

Area $= \int_{-2\pi}^{2\pi} y(x)dx = -\int_0^{2\pi} r(1 - \cos\theta) \cdot \frac{d}{d\theta}(r(\theta - \sin\theta))d\theta$

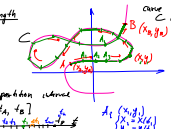
$= \int_0^{2\pi} r(1 - \cos\theta) \cdot r(1 - \cos\theta)d\theta = \int_0^{2\pi} r^2(1 - \cos\theta)^2 d\theta$

$= r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta = r^2 \cdot 3 \cdot 2\pi = 3\pi r^2$

$\int_0^{2\pi} \sin x dx = 0$

$\int_0^{2\pi} \cos^n x dx = \ldots$

Arc Length



Curve $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t_a \le t \le t_b$

Q. Find length of C between A and B

$\begin{cases} x(t_a) = x_a \\ y(t_a) = y_a \end{cases}$ $\begin{cases} x(t_b) = x_b \\ y(t_b) = y_b \end{cases}$

First, partition interval $[t_a, t_b]$

$t_a = t_0 < t_1 < t_2 < t_3 \le t_{n-1} \le t_n = t_b$

$\Delta t_1 = t_1 - t_0$

$\Delta t_i = t_i - t_{i-1}$

$A_i(x_i, y_i)$ $A_{i+1}(x_{i+1}, y_{i+1})$

$|A_i A_{i+1}| = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$

Δx_i Δy_i

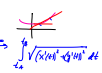
Second, calculate the following sum

$\Delta t_i \begin{cases} x_i, y_i \\ x_i = x(t_i) \\ y_i = y(t_i) \end{cases}$

$S_n = \sum_{i=0}^{n-1} |A_i A_{i+1}|$ - length of zig-zag path

Third, take the limit

$S_n \xrightarrow{n \to \infty} L$



$S_n = \sum_{i=0}^{n-1}\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=0}^{n-1}\sqrt{(x'(t_i))^2 + (y'(t_i))^2}\,\Delta t_i \xrightarrow{n \to \infty} \int_{t_a}^{t_b}\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$\Delta x_i = x'(t_i)\Delta t_i + \ldots$

$\Delta y_i = y'(t_i)\Delta t_i + \ldots$

$\left(\begin{array}{l} dx = x'(t)dt \\ dy = y'(t)dt \end{array}\right)$

Length of the arc between points A and B is

$L = \int_{t_a}^{t_b}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex circle $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $0 \le \theta \le 2\pi$

$L = \int_0^{2\pi}\sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\,d\theta = r\int_0^{2\pi}\sqrt{\sin^2\theta + \cos^2\theta}\,d\theta = 2\pi r$

Ex. Cycloid $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $\begin{cases} x' = r(1 - \cos\theta) \\ y' = r\sin\theta \end{cases}$

$0 \le \theta \le 2\pi$

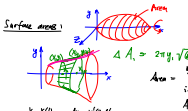
$L = \int_0^{2\pi}\sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\,d\theta$

$= r\int_0^{2\pi}\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta$

$\left(1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right)$ $= r\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\,d\theta = r\int_0^{2\pi}\sqrt{4\sin^2\frac{\theta}{2}}\,d\theta$

$= r\int_0^{2\pi} 2\sin\frac{\theta}{2}\,d\theta = 2r\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 2r\left[2 - (-2)\right] = 8r$

Surface areas



$\Delta A_i = 2\pi y_i\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $\begin{cases} dx = x'(t)dt \\ dy = y'(t)dt \end{cases}$

$t_a \le t \le t_b$

$\text{Area} = \sum_{i=0}^{n-1}\Delta A_i = \sum_{i=0}^{n-1} 2\pi y_i\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

As we take to limit $n \to \infty$

$\text{Area} = \int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$ $\begin{cases} x' = r(1 - \cos\theta) \\ y' = r\sin\theta \end{cases}$

$A = \int_0^{2\pi}\pi r(1 - \cos\theta)\,2r\sin\frac{\theta}{2}\,d\theta = 4\pi r^2\int_0^{2\pi} 2\sin\frac{\theta}{2}\sin^2\frac{\theta}{2}\,d\theta$

$= 8\pi r^2\int_0^{2\pi}\sin^3\frac{\theta}{2}\,d\theta = 8\pi r^2 \cdot \frac{1}{2} \cdot 2\pi = 8\pi^2 r^2$

$\int_0^{2\pi}\sin^3\frac{\theta}{2}$

$\Delta A_i = 2\pi x_i \Delta S_i = 2\pi x_i\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

$A = \int_{t_a}^{t_b} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Lecture 4 (10.5-10.8)

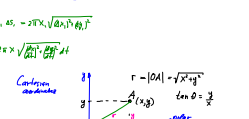
Polar coordinates

Cartesian coordinates

$(x, y) \longrightarrow (r, \theta)$

$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



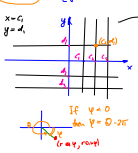
$r = |OA| = \sqrt{x^2 + y^2}$

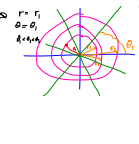
$\tan\theta = \frac{y}{x}$

$[0 \le \theta < 2\pi]$ then $\tan\theta$ is undefined

$x = C_1$

$y = d_1$





If $\varphi = 0$

then $\varphi = 0 - 2\pi$

$(r \cos\varphi, r\sin\varphi)$

Parametric curves in polar coordinates

Define $\begin{cases} x = x(t) \\ y = y(t) \end{cases} \longrightarrow \begin{cases} r = r(\theta) \\ \theta = \theta(t) \end{cases}$

As a particular case

$r = r(\theta)$

Ex $r = 5$