A Matrix Expander Chernoff Bound

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Vanilla Chernoff Bound

Thm [Hoeffding, Chernoff]. If $X_1, \ldots, X_k$ are independent mean zero random variables with $|X_i| \leq 1$ then

$$\mathbb{P}\left[\left|\frac{1}{k}\sum_i X_i\right| \geq \epsilon\right] \leq 2\exp(-k\epsilon^2/4)$$

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Two Extensions:

1. **Dependent** Random Variables
2. Sums of random **matrices**

Expander Chernoff Bound [AKS'87, G'94]

Thm[Gillman'94]: Suppose $G = (V, E)$ is a regular graph with transition matrix P which has second eigenvalue λ. Let $f: V \to \mathbb{R}$ be a function with $|f(v)| \leq 1$ and $\sum_v f(v) = 0$.

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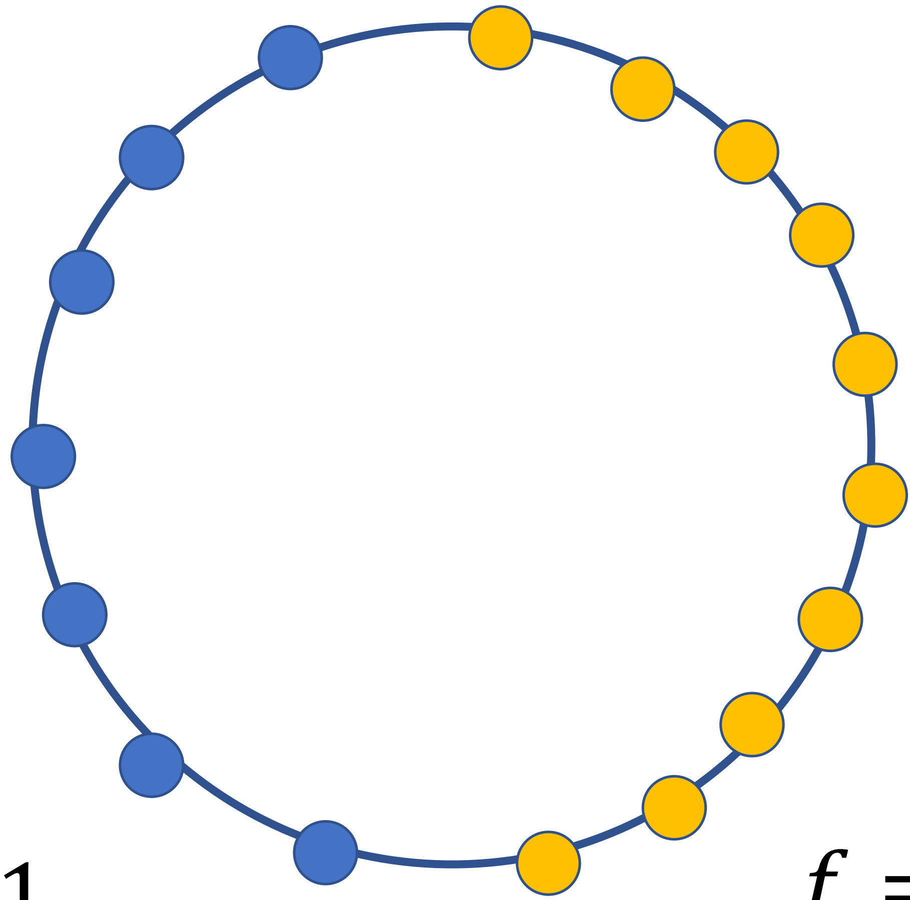
Then, if $v_1, \ldots, v_k$ is a stationary random walk:

$$\mathbb{P}\left[\left|\frac{1}{k}\sum_i f(v_i)\right| \geq \epsilon\right] \leq 2\exp(-c(1-\lambda)k\epsilon^2)$$

Implies walk of length $k \approx (1-\lambda)^{-1}$ concentrates around mean.

Intuition for Dependence on Spectral Gap

$$1 - \lambda = \frac{1}{n^2}$$



$f = +1$

$f = -1$

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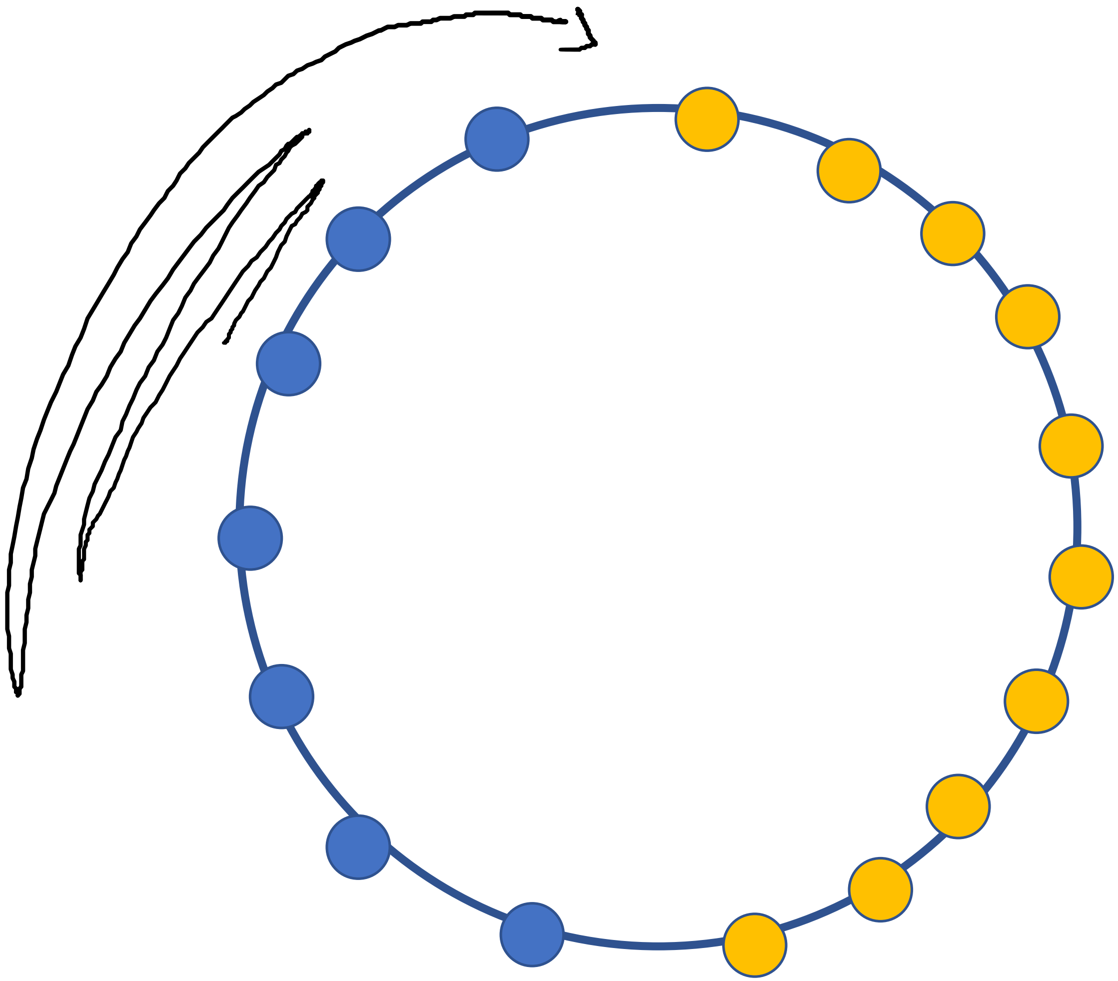

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Typical random walk takes $\Omega(n^2)$ steps to see both ± 1.

Derandomization Motivation

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If G is $3-$regular with constant λ, walk needs: $\log n + k \log(3)$ bits.

When $k = O(\log n)$ reduces randomness *quadratically*.
Can completely derandomize in polynomial time.

Matrix Chernoff Bound

Thm [Rudelson'97, **Ahlswede-Winter'02**, Oliveira'08, Tropp'11…].
If $X_1, \dots, X_k$ are independent mean zero random $d \times d$ Hermitian matrices with $||X_i|| \leq 1$ then

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Factor d is tight because of the diagonal case.

Very generic bound (no independence assumptions on the entries). Many applications + martingale extensions (see Tropp).

Matrix Expander Chernoff Bound?

Conj[Wigderson-Xiao'05]: Suppose $G = (V, E)$ is a regular graph with transition matrix P which has second eigenvalue λ. Let $f: V \to \mathbb{C}^{d\times d}$ be a function with $||f(v)|| \leq 1$ and $\sum_v f(v) = 0$. Then, if $v_1, \ldots, v_k$ is a stationary random walk:

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Motivated by derandomized Alon-Roichman theorem.

Main Theorem

Thm. Suppose $G = (V, E)$ is a regular graph with transition matrix P which has second eigenvalue λ. Let $f: V \to \mathbb{C}^{d\times d}$ be a function with $||f(v)|| \leq 1$ and $\sum_v f(v) = 0$.
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Gives black-box derandomization of any application of matrix Chernoff

1. Proof of Chernoff: reduction to mgf

Thm [Hoeffding, Chernoff]. If $X_1, \ldots, X_k$ are independent mean zero random variables with $|X_i| \leq 1$ then

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$$\mathbb{P}\left[\sum_i X_i \geq k\epsilon\right] \leq e^{-tk\epsilon}\,\mathbb{E}\exp\left(t\sum_i X_i\right)$$

Markov

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Indep.

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$$\leq e^{-tk\epsilon}(1 + t\mathbb{E}X_i + t^2)^k$$

Bounded

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Mean zero

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$$\leq e^{-tk\epsilon}(1 + \qquad + t^2)^k \leq e^{-tk\epsilon+kt^2} \leq \mathrm{e}^{-\frac{k\epsilon^2}{4}}$$

□

2. Proof of Expander Chernoff

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Issue: $\mathbb{E}\exp(\sum_i f(v_i)) \neq \prod_i \mathbb{E}\exp(tf(v_i))$

How to control the mgf without independence?

Step 1: Write mgf as quadratic form

$$\mathbb{E}e^{t\sum_{i\le k} f(v_i)}$$

$$= \sum_{i_0,\dots,i_k\in V} \mathbb{P}(v_0 = i_0)P(i_0, i_1)\dots P(i_{k-1}, i_k)\exp\left(t\sum_{1\le j\le k} f(i_j)\right)$$

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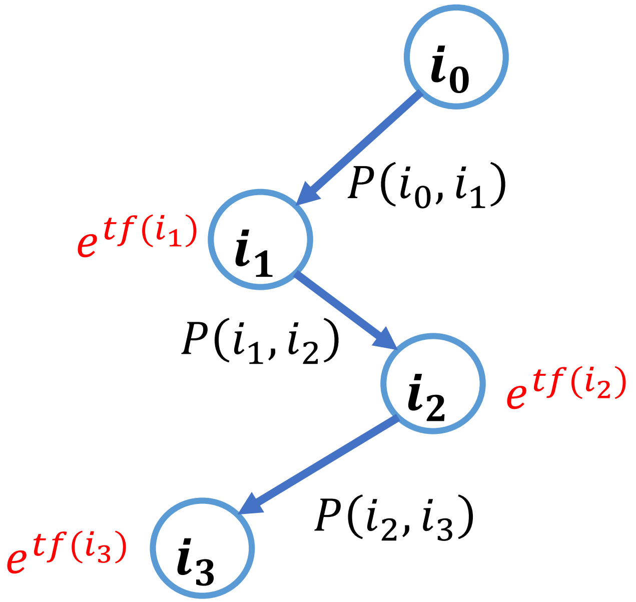

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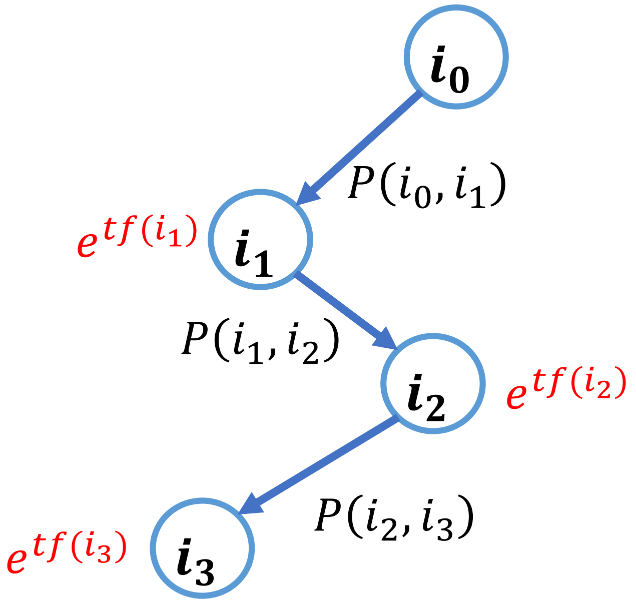
$$= \langle u, (EP)^k u\rangle \text{ where } u = \left(\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}\right).$$

$$E = \begin{bmatrix} e^{tf(1)} & & \\ & \dots & \\ & & e^{tf(n)} \end{bmatrix}$$



Step 2: Bound quadratic form

Goal: Show $\langle u, (EP)^k u \rangle \leq \exp(c_\lambda k t^2)$

Observe: $||P - J|| \leq \lambda$ where $J =$ complete graph with self loops

So for small λ, should have $\langle u, (EP)^k u \rangle \approx \langle u, (EJ)^k u \rangle$

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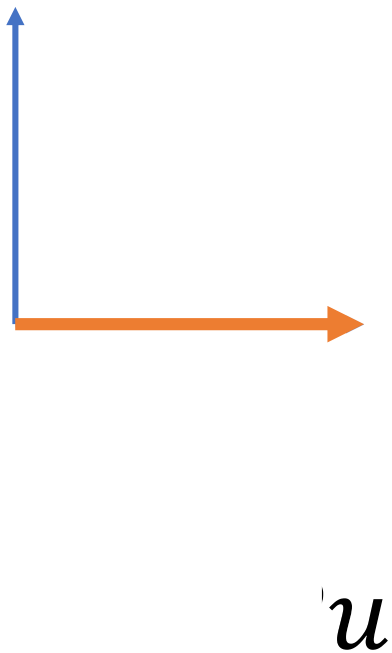
Approach 2. (Healy'08) track projection of iterates along u.

Simplest case: $\lambda = 0$



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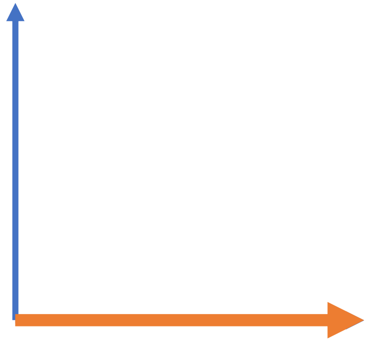

Pu

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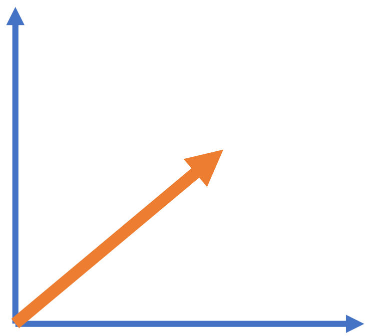

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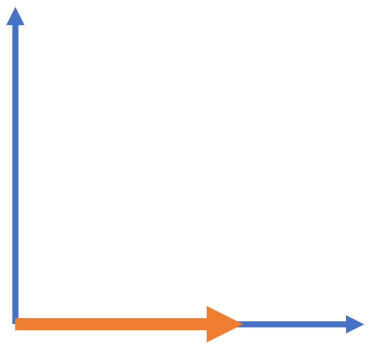

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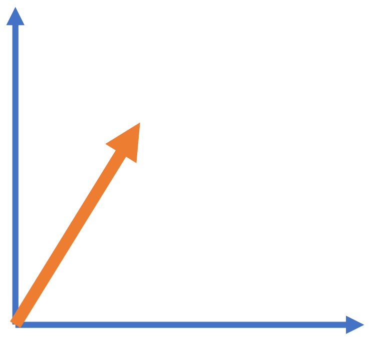

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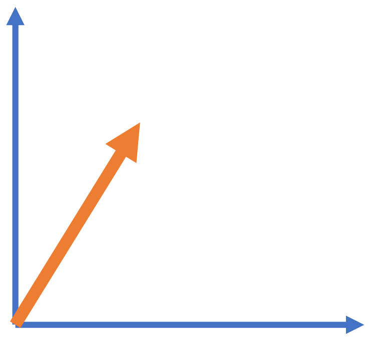
Observations

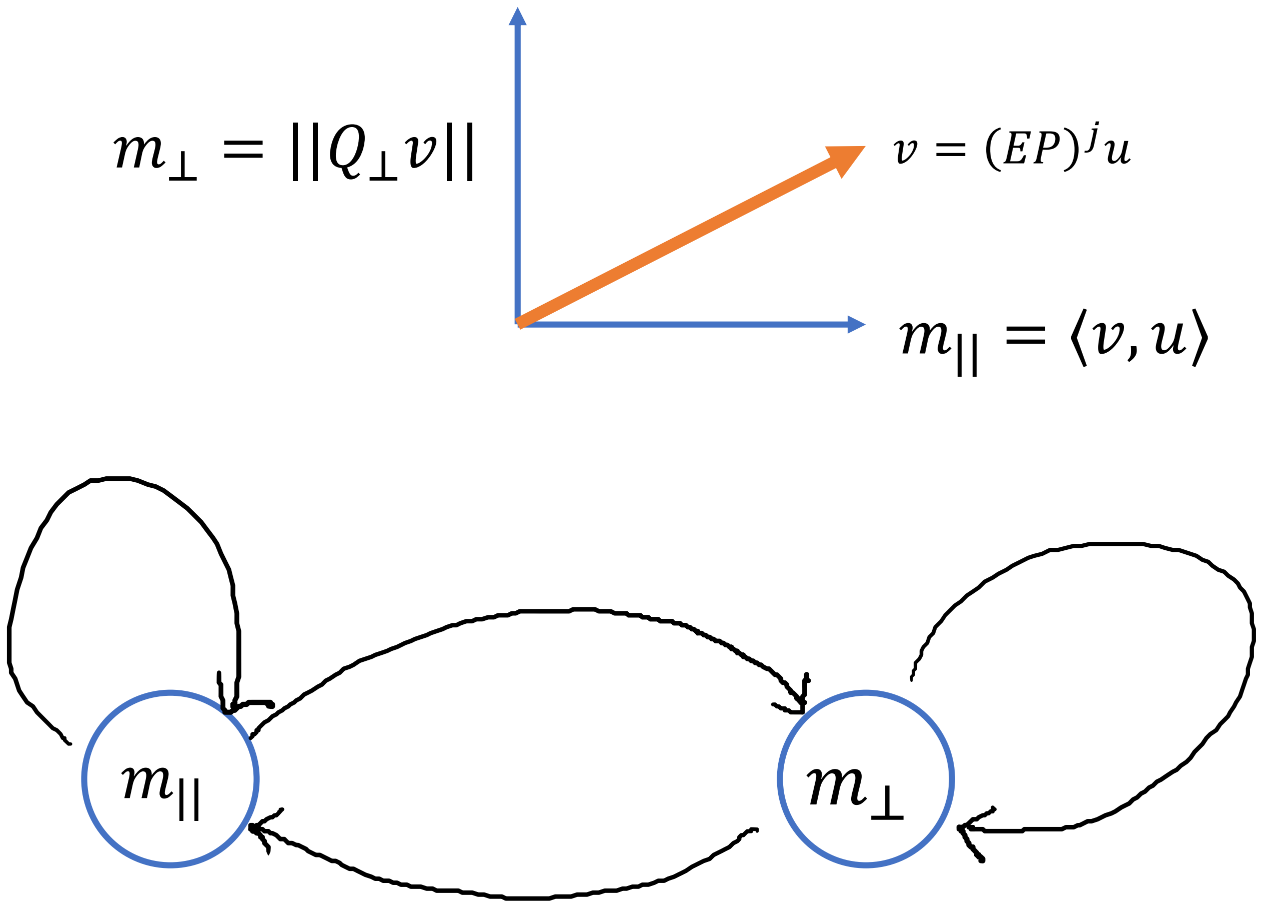


$$EPEPu$$

Observe: P shrinks every vector orthogonal to u by λ.

$\langle u, Eu \rangle = \frac{1}{\mathrm{n}} \sum_{v \in \mathrm{V}} e^{tf(v)} = 1 + t^2$ by mean zero condition.

A small dynamical system



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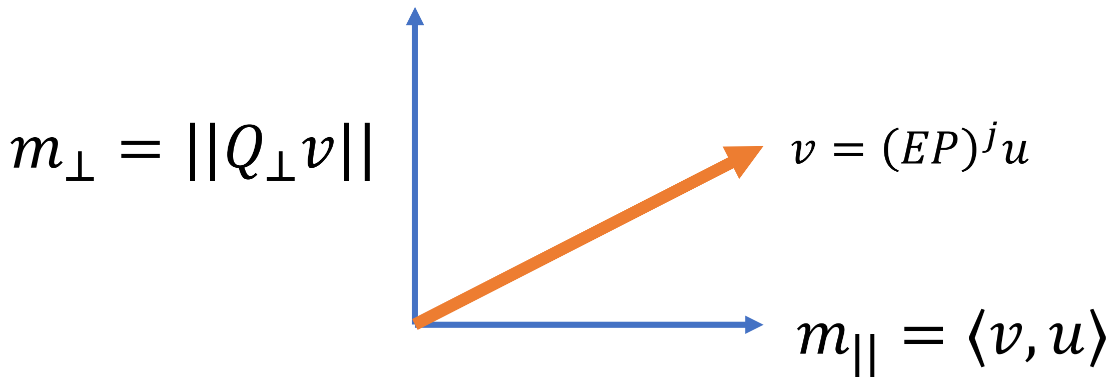

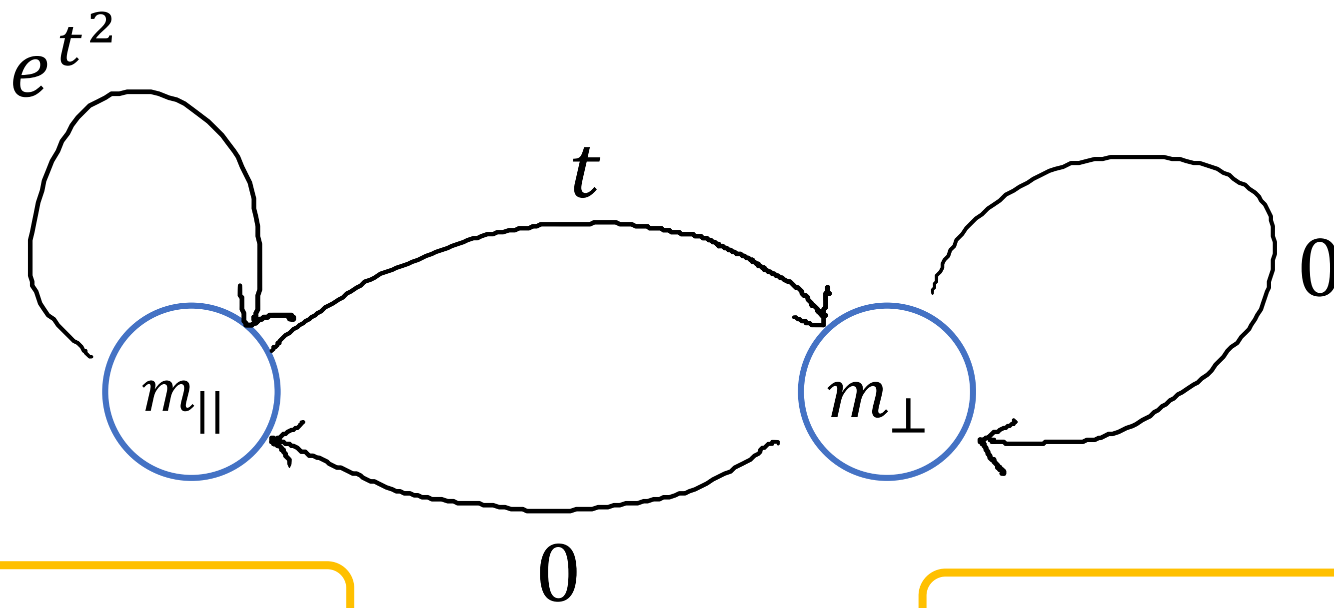

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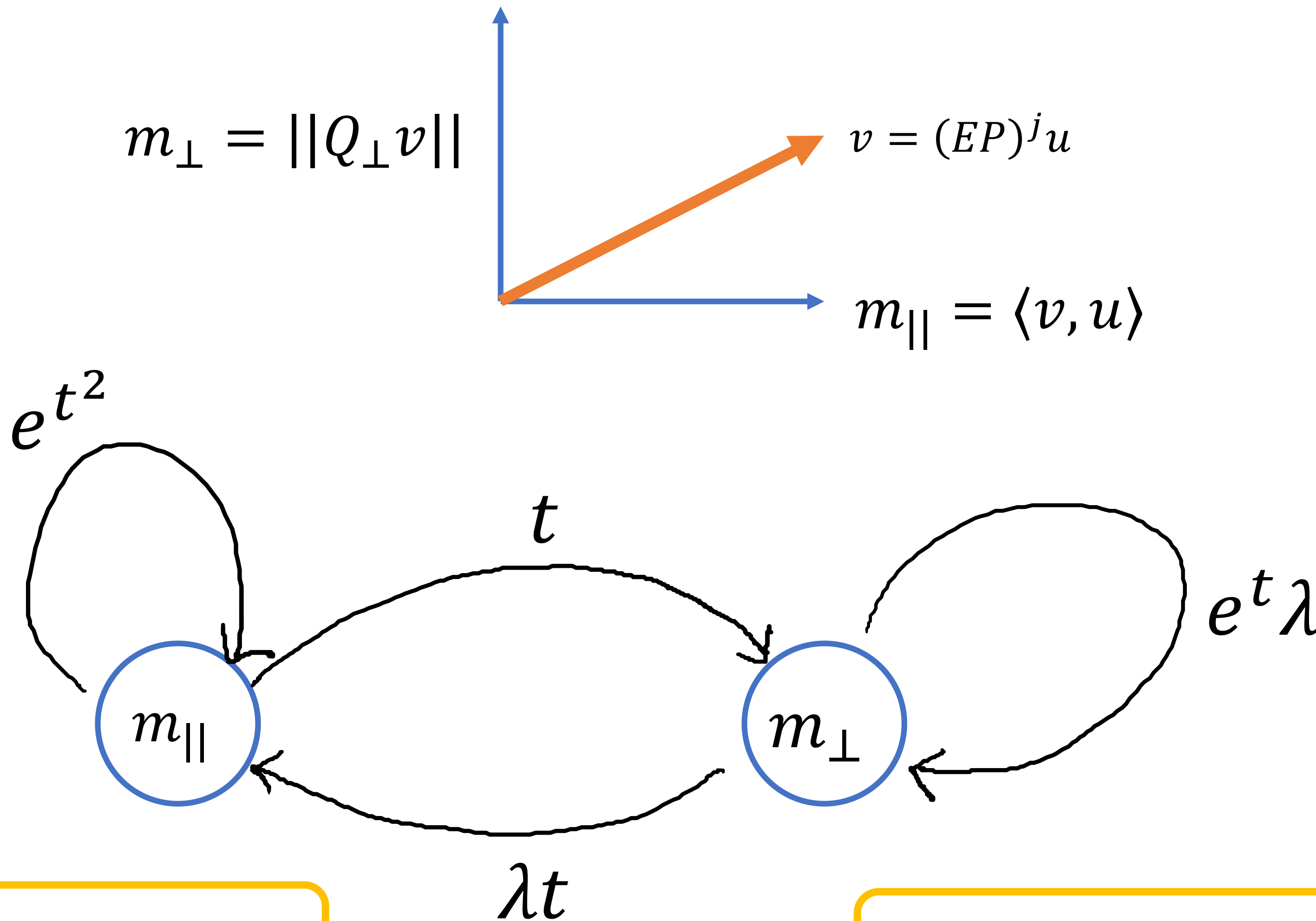

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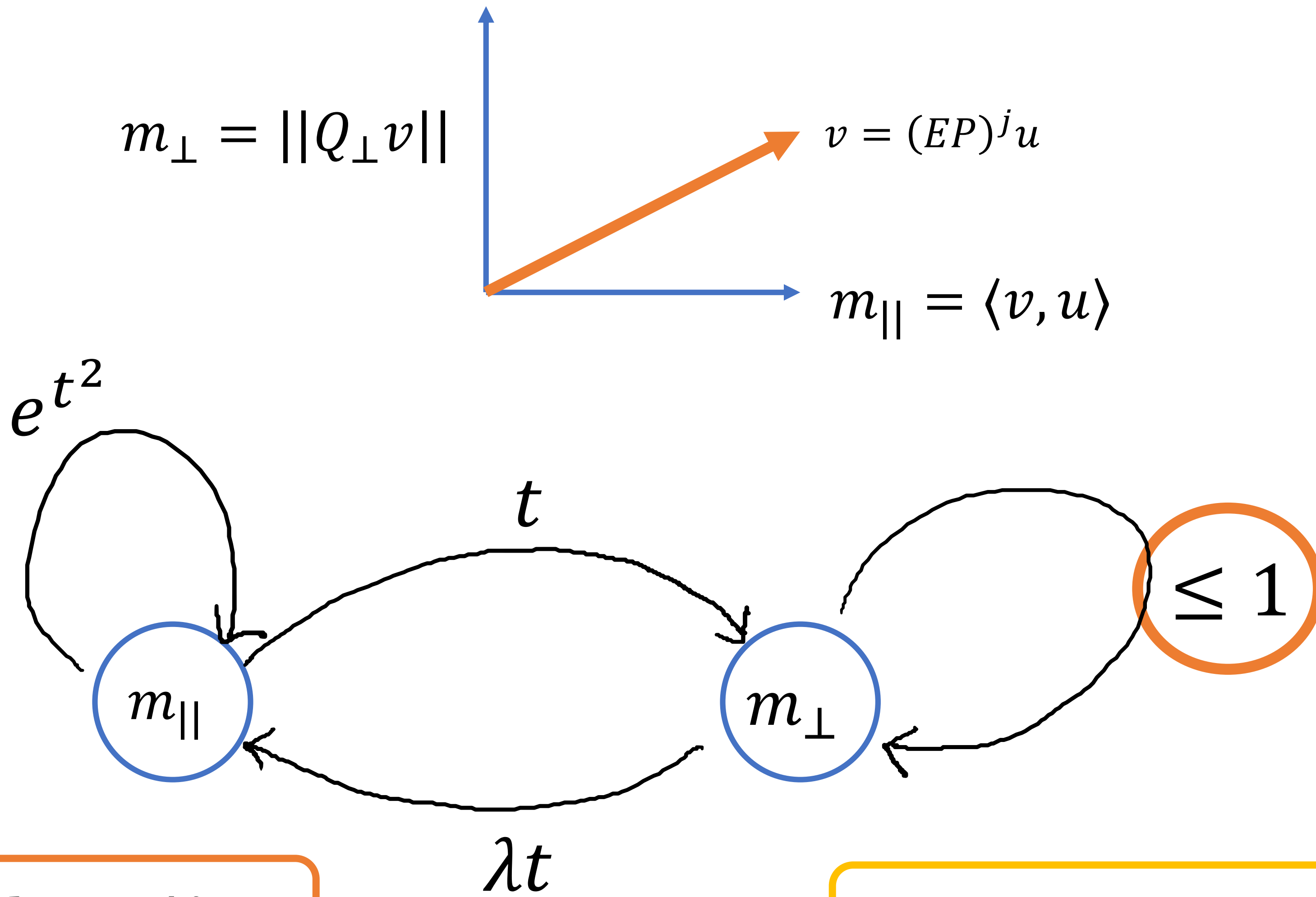

$t \leq \log 1/\lambda$

$v \curvearrowright EPv$

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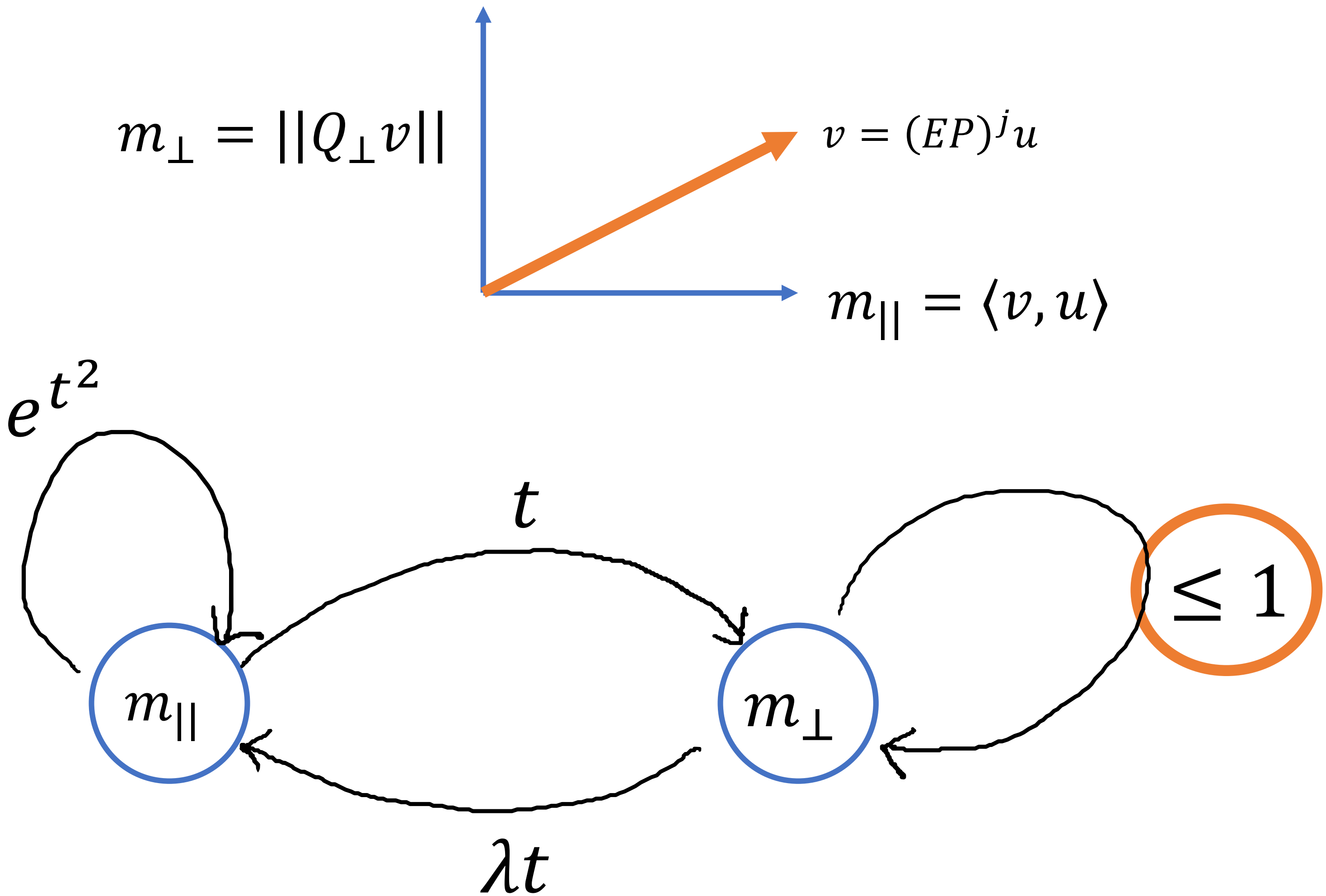

Any mass that leaves gets shrunk by λ

Analyzing dynamical system gives

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Generalization to Matrices?

Setup: $f: V \to \mathbb{C}^{d\times d}$, random walk $v_1, \dots, v_k$.

Goal:

$$\mathbb{E}Tr\left[\exp\left(t\sum_i f(v_i)\right)\right] \leq d \cdot \exp(ckt^2)$$

where e^A is defined as a power series.

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Main Issue: $\exp(A + B) \neq \exp(A)\exp(B)$ unless $[A, B] = 0$

can't express exp(sum) as iterated product.

The Golden-Thompson Inequality

Partial Workaround [Golden-Thompson'65]:

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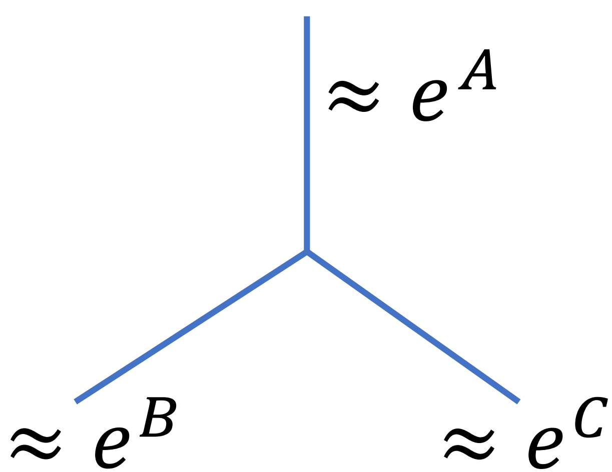

$$Tr(e^{A+B+C}) > 0 > Tr(\mathrm{e}^{\mathrm{A}}\mathrm{e}^{\mathrm{B}}\mathrm{e}^{\mathrm{C}})$$

Key Ingredient

[Sutter-Berta-Tomamichel'16] If $A_1, \ldots, A_k$ are Hermitian, then

$$\log Tr(e^{A_1 + \ldots + A_k})$$
$$\leq \int d\beta(b) \log Tr\left[\left(e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}}\right)\left(e^{\frac{A_1(1+ib)}{2}} \ldots e^{\frac{A_k(1+ib)}{2}}\right)^*\right]$$

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1. Matrix on RHS is always PSD.
2. **Average-case** inequality: $e^{A_i/2}$ are conjugated by unitaries.
3. Implies Lieb's concavity, triple-matrix, ALT, and more.

Proof of SBT: Lie-Trotter Formula

$$e^{A+B+C} = \lim_{\theta \to 0^+} \left(e^{\theta A} e^{\theta B} e^{\theta C} \right)^{1/\theta}$$

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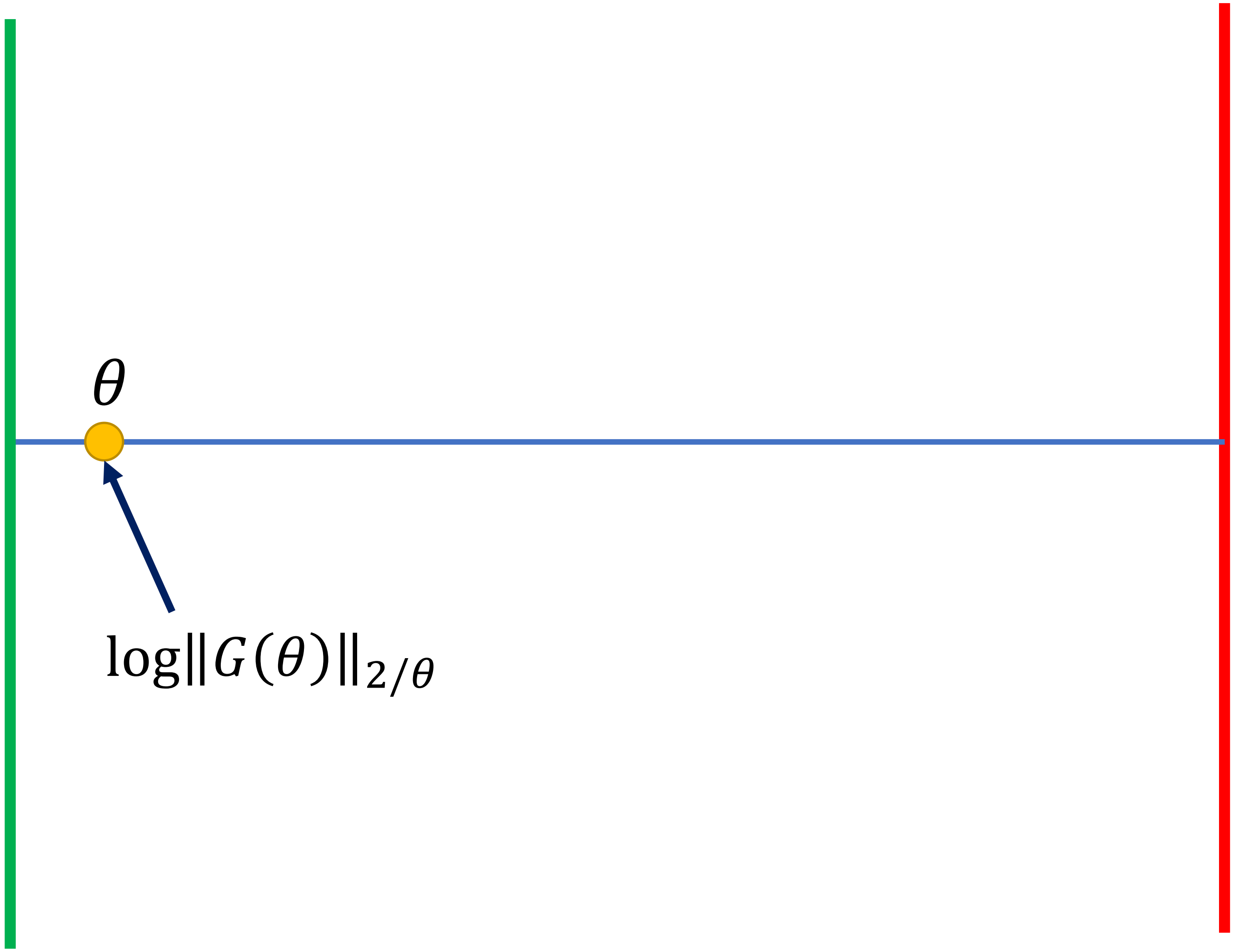
$$\log Tr\, e^{A+B+C} = \lim_{\theta \to 0^+} 2\log ||G(\theta)||_{2/\theta} / \theta$$

For $G(z) \coloneqq e^{\frac{zA}{2}} e^{\frac{zB}{2}}\, e^{\frac{zC}{2}}$

Complex Interpolation (Stein-Hirschman)



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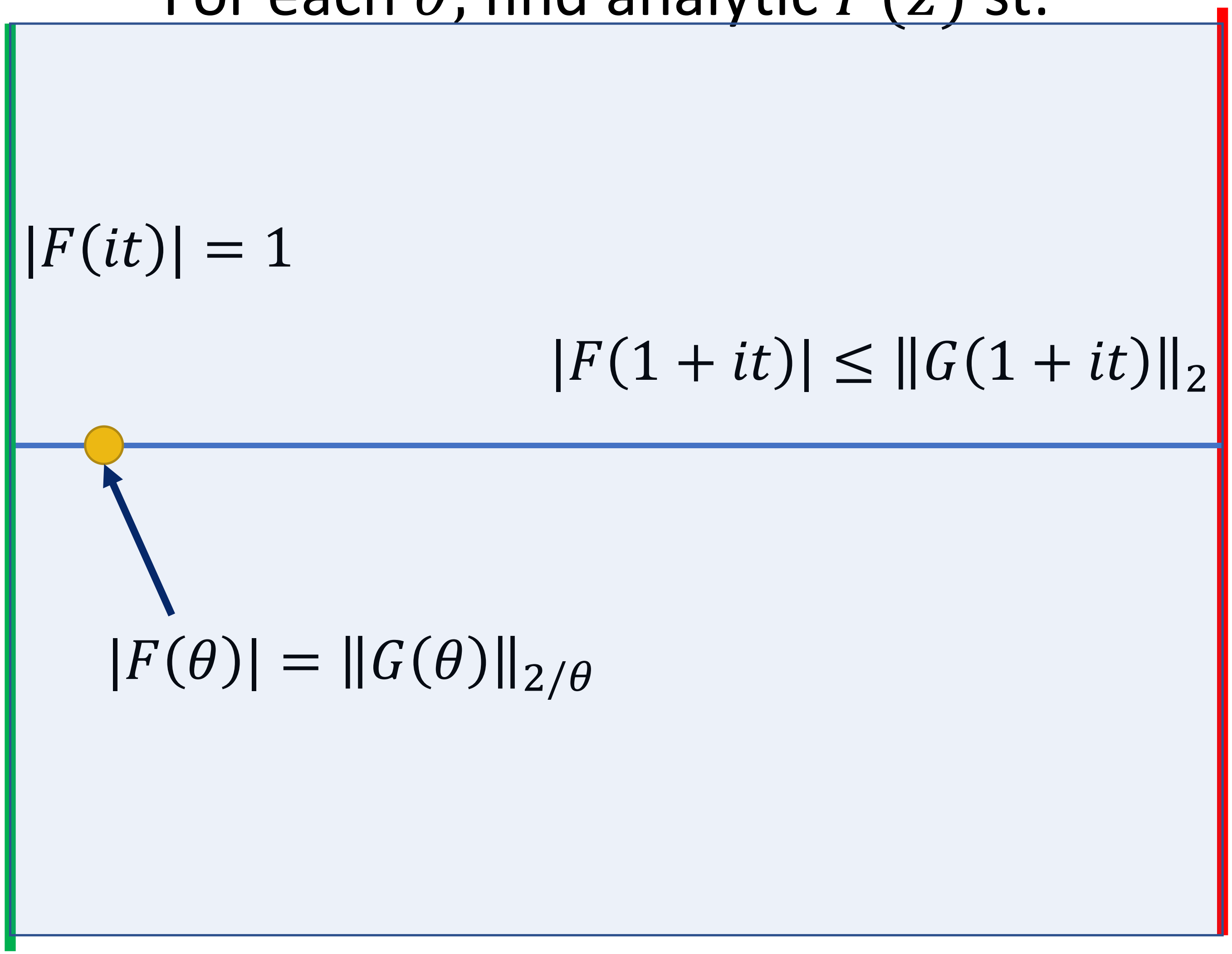
For each θ, find analytic $F(z)$ st:



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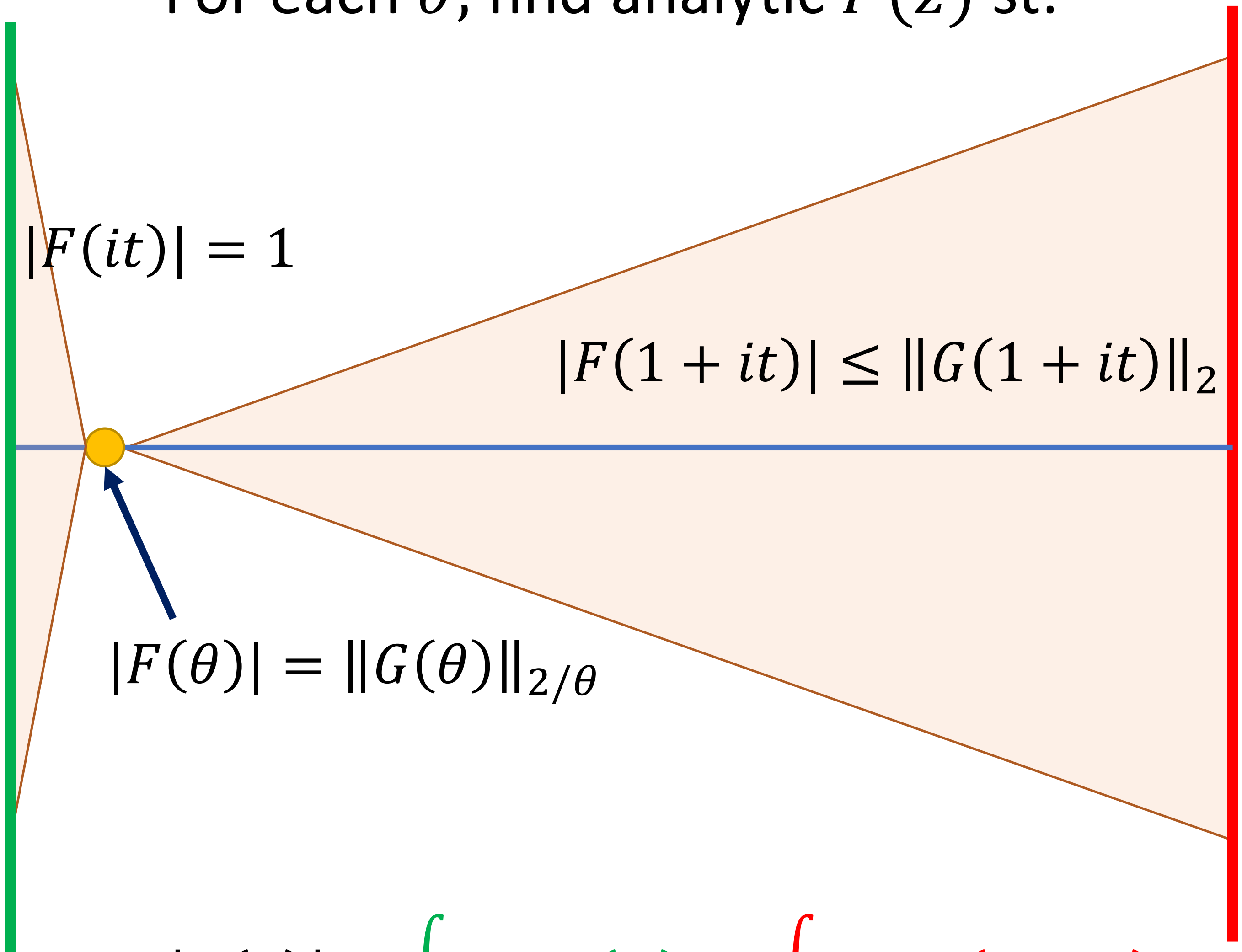

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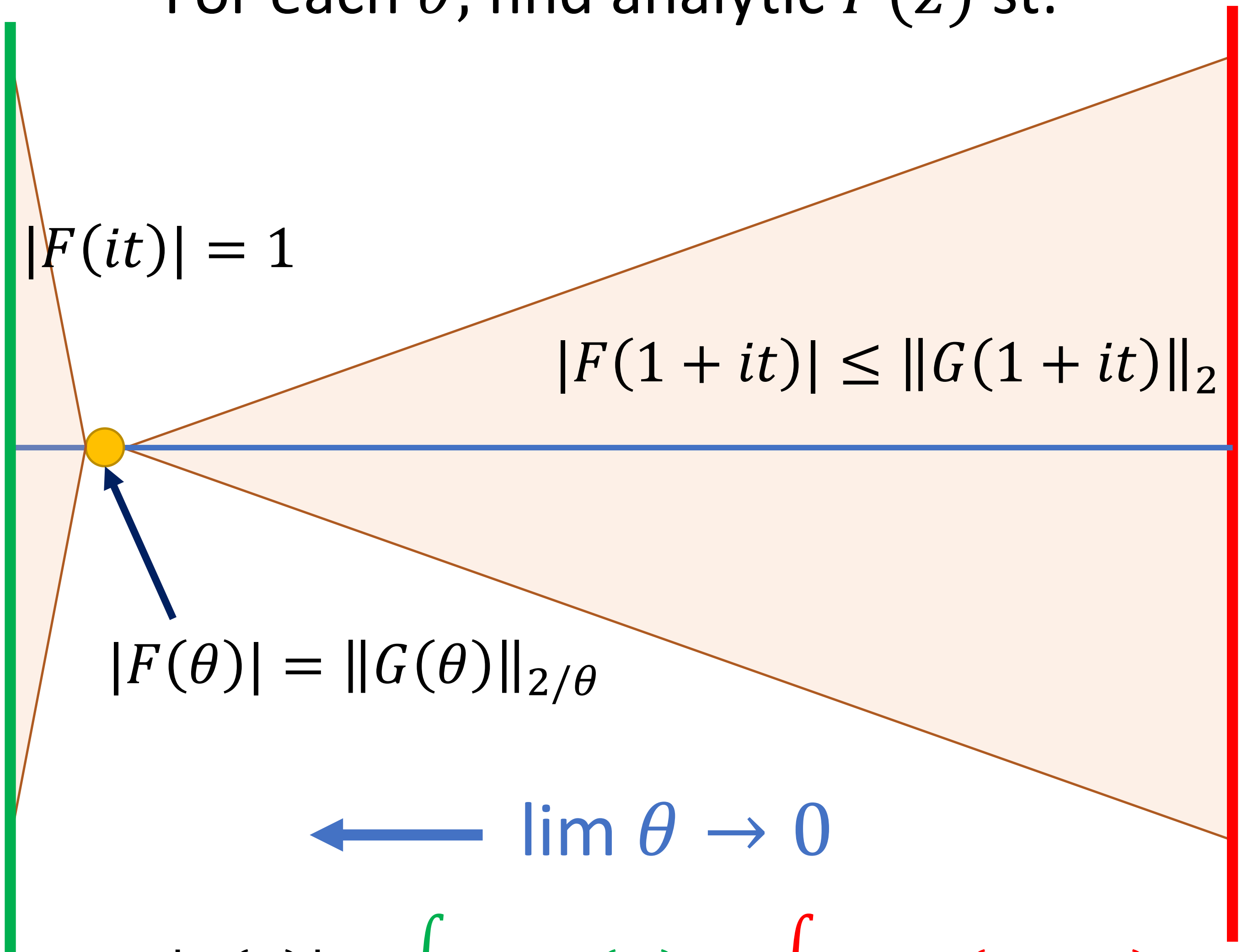

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[Sutter-Berta-Tomamichel'16] If $A_1, \dots, A_k$ are Hermitian, then

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$$\leq \int d\beta(b) \log Tr\left[\left(e^{\frac{A_1(1+ib)}{2}} \dots e^{\frac{A_k(1+ib)}{2}}\right)\left(e^{\frac{A_1(1+ib)}{2}} \dots e^{\frac{A_k(1+ib)}{2}}\right)^*\right]$$

where $\beta(b)$ is an explicit probability density on $\mathbb{R}$.

Issue. SBT involves integration over unbounded region, bad for Taylor expansion.

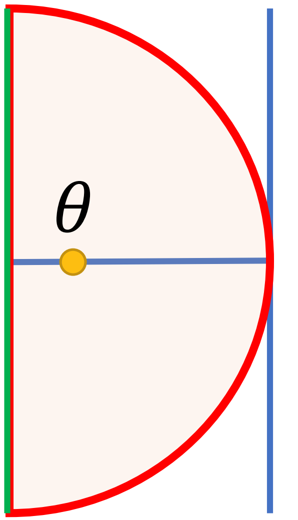
Bounded Modification of SBT

Solution. Prove bounded version of SBT by replacing strip with half-disk.

[Thm] If $A_1, \ldots, A_k$ are Hermitian, then

$$\log Tr(e^{A_1 + \ldots + A_k})$$
$$\leq \int d\beta(b) \log Tr\left[\left(e^{\frac{A_1 e^{ib}}{2}} \ldots e^{\frac{A_k e^{ib}}{2}}\right)\left(e^{\frac{A_1 e^{ib}}{2}} \ldots e^{\frac{A_k e^{ib}}{2}}\right)^*\right]$$

where $\beta(b)$ is an explicit probability density on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Proof. Analytic $F(z)$ + Poisson Kernel + Riemann map.

Handling Two-sided Products

Issue. Two-sided rather than one-sided products:

$$Tr\left[\left(e^{\frac{tf(v_1)e^{ib}}{2}} \ldots e^{\frac{tf(v_k)e^{ib}}{2}}\left(e^{\frac{tf(v_1)e^{ib}}{2}} \ldots e^{\frac{tf(v_k)e^{ib}}{2}}\right)^*\right)\right]$$

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Solution.

Encode as one-sided product by using $Tr(AXB) = (A \otimes B^T)vec(X)$:

$$\langle e^{\frac{tf(v_1)e^{ib}}{2}} \otimes e^{\frac{tf(v_1)^{*T}e^{ib}}{2}} \dots e^{\frac{tf(v_k)^{*T}e^{-ib}}{2}} \otimes e^{\frac{tf(v_k)^{*T}e^{-ib}}{2}} \mathrm{vec}(\mathrm{I_d}), \mathrm{vec}(\mathrm{I_d})$$

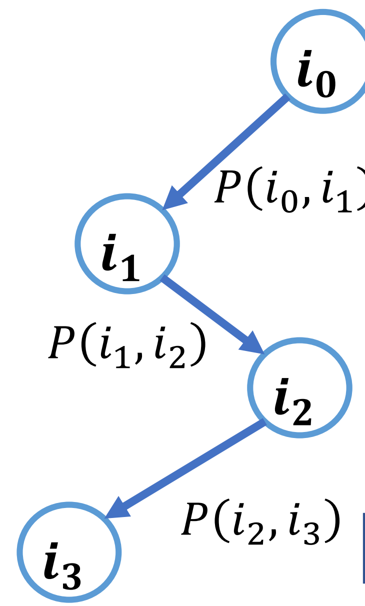
Finishing the Proof

Carry out a version of Healy's argument with $P \otimes I_{d^2}$ and:

$$E = \begin{bmatrix} e^{\frac{tf(1)e^{ib}}{2}} \otimes e^{\frac{tf(1)^{*T}e^{ib}}{2}} & & \\ & \cdots & \\ & & e^{\frac{tf(n)e^{ib}}{2}} \otimes e^{\frac{tf(n)^{*T}e^{-ib}}{2}} \end{bmatrix}$$

And $vec(I_d) \otimes u$ instead of u.

This leads to the additional d factor.



$\square$

Main Theorem

Thm. Suppose $G = (V, E)$ is a regular graph with transition matrix P which has second eigenvalue λ. Let $f: V \to \mathbb{C}^{d\times d}$ be a function with $||f(v)|| \leq 1$ and $\sum_v f(v) = 0$.
Then, if $v_1, \dots, v_k$ is a stationary random walk:

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i f(v_i)\right\| \geq \epsilon\right] \leq 2d\exp(-c(1-\lambda)k\epsilon^2)$$

Open Questions

Other matrix concentration inequalities

(multiplicative, low-rank, moments)

Other Banach spaces

(Schatten norms)

More applications of complex interpolation