Girth, Expansion, and Localization of Graph Eigenfunctions

Nikhil Srivastava
w/ Noga Alon (Princeton) and Shirshendu Ganguly (Berkeley)

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Setup

Undirected $d + 1$-regular graph G on n vertices.

Adjacency matrix A has eigenvalues

$$d + 1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

?

cuts [Alon-Milman-Chung]

Coloring [Hoffman]

Q. What is the combinatorial meaning of the interior eigenvectors?

A Warmup

Observation. If $Av = \lambda v$ and v is supported on k vertices, then

$$girth(A) \leq 4\log_d(k)$$

length of shortest cycle

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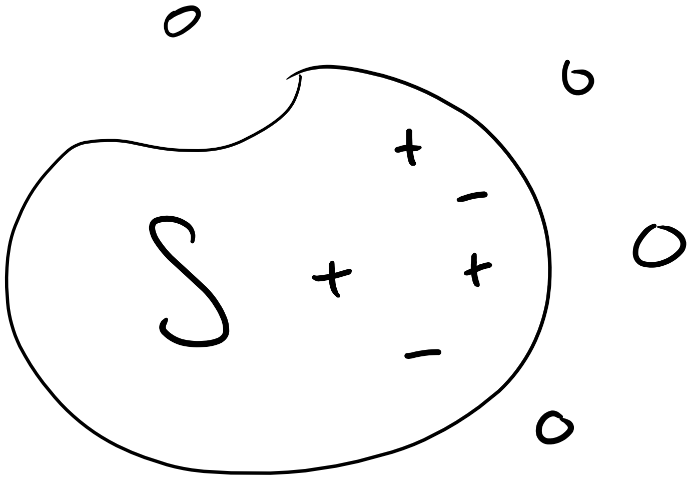
Proof.

G



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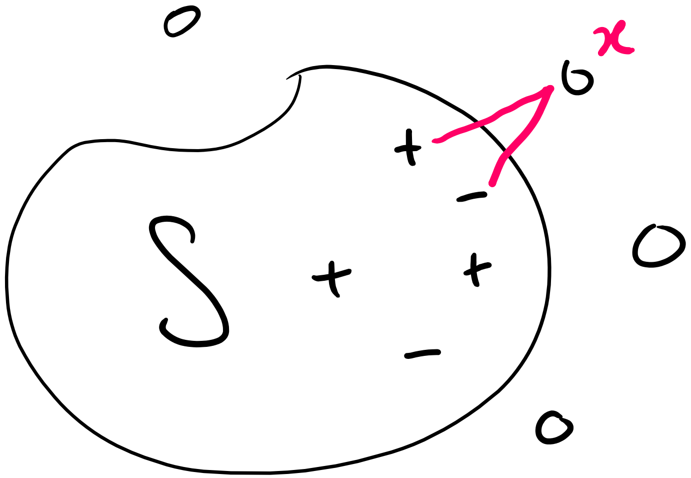

$$0 = v(x) = \sum_{y \sim x} v(y)$$

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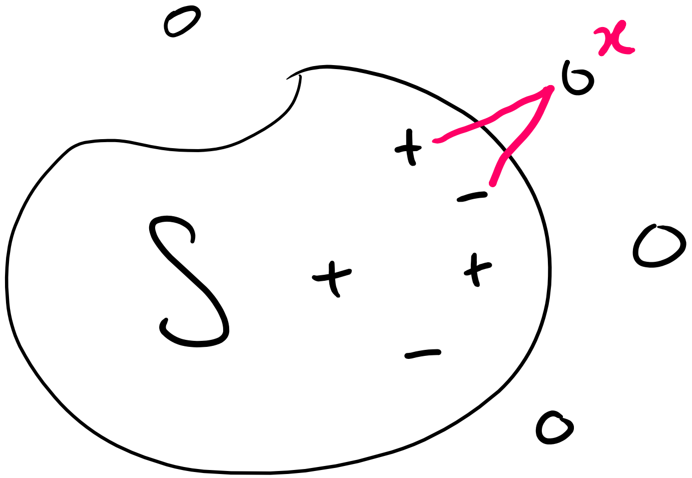

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Every vertex adjacent to S has at least two nbrs in S.

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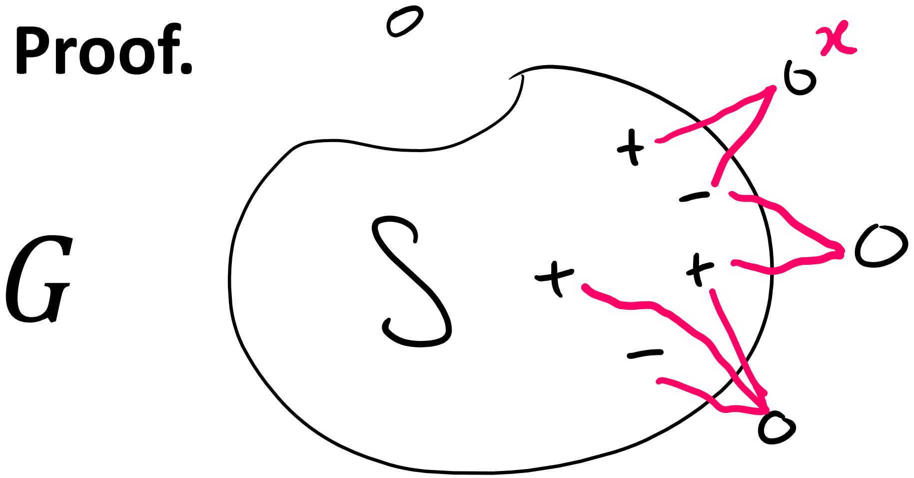

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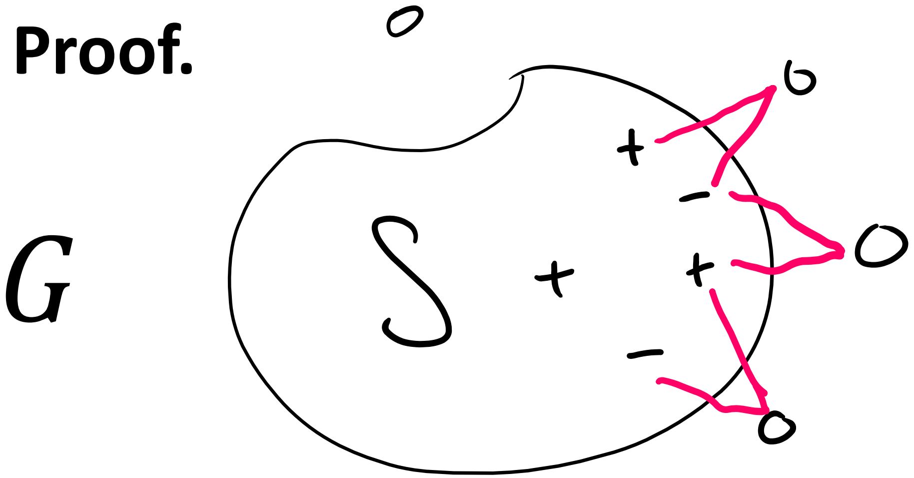

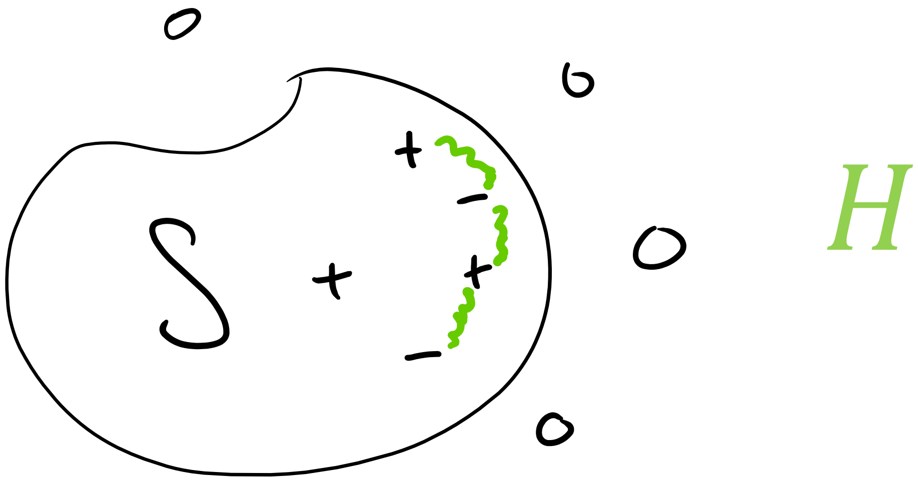

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Replace excursions of length 2 by new edges.

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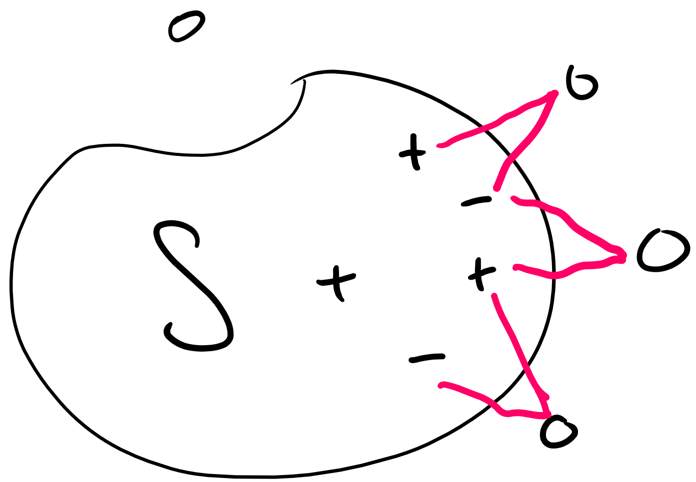

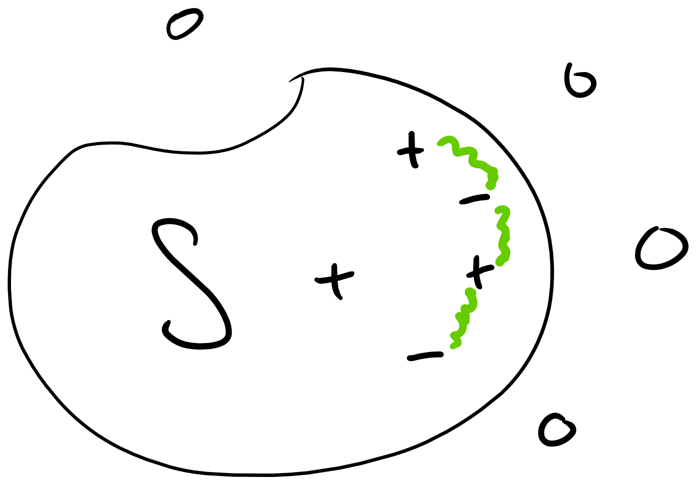

H

$$mindeg(H) \geq d + 1 \Rightarrow$$
$$girth(H) \leq 2\log_d(k)$$

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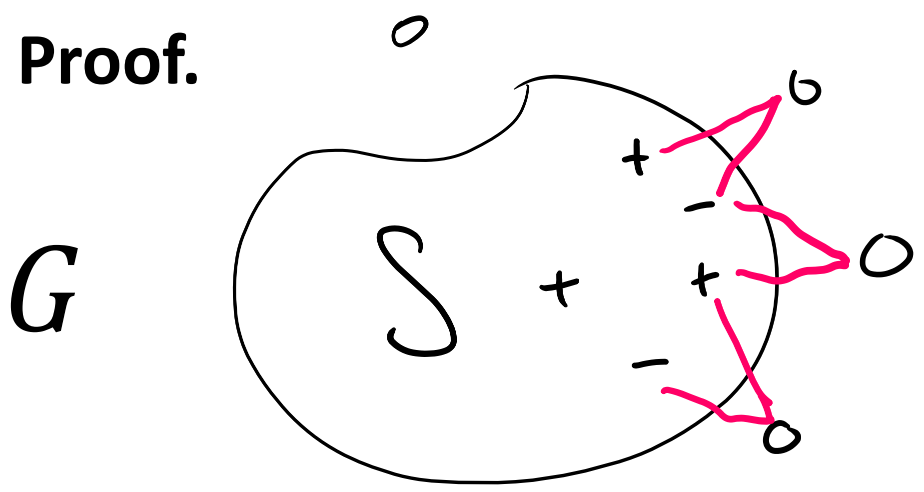

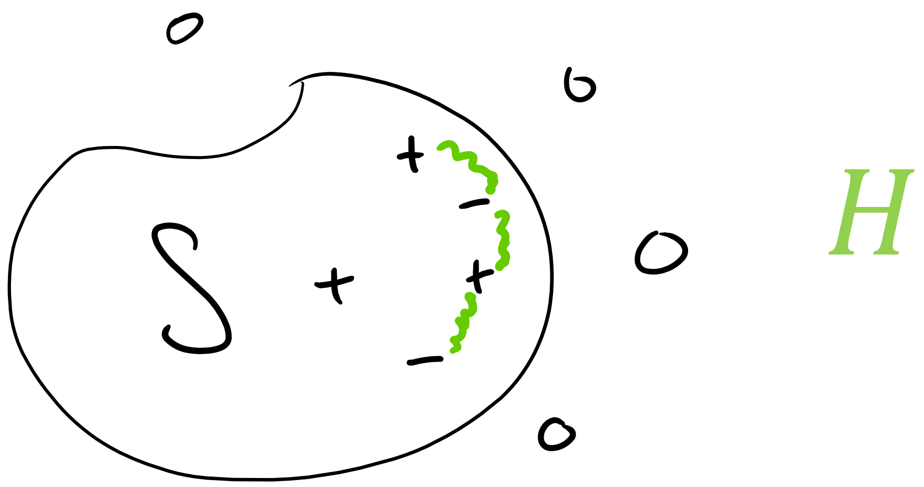

$$girth(G) \le 4\log_d(k)$$

$$mindeg(H) \ge d+1 \Rightarrow girth(H) \le 2\log_d(k)$$

□

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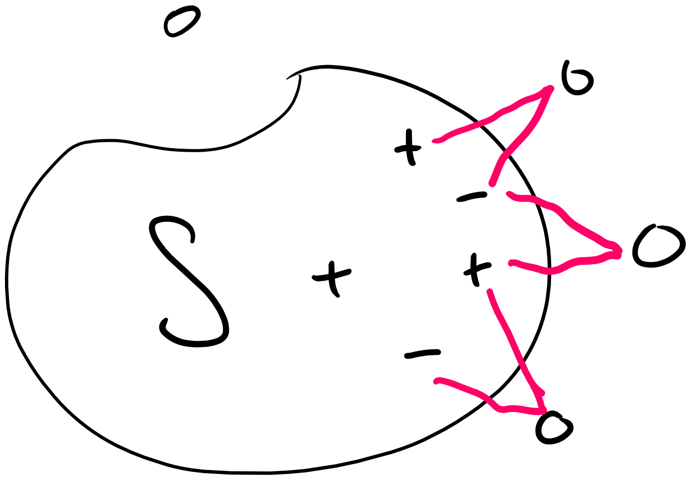

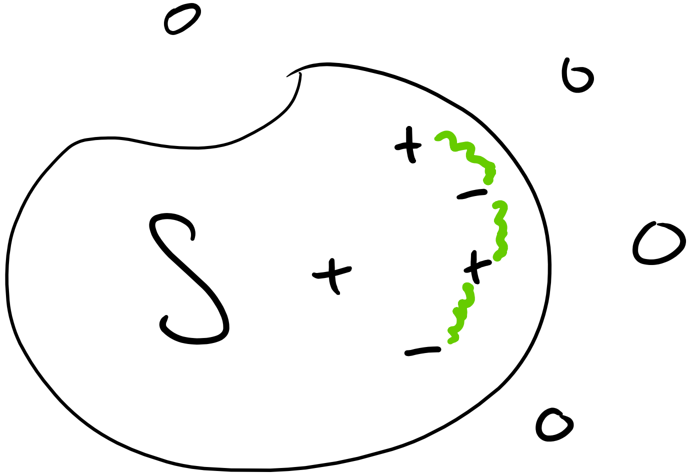

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$$girth(G) \leq 4\log_d(k)$$

$$k \geq d^{g/4}$$

□

Localization and Delocalization

Defn. A unit vector v is (ϵ, k) $-$**delocalized** if for all subsets $S \subset [n]$:

$$||v_S||_2^2 \geq \epsilon \Rightarrow |S| > k$$

Otherwise it is (ϵ, k)-**localized**, i.e., $||v_S||_2^2 \geq \epsilon$ for some $|S| \leq k$.

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e.g. k $-$sparse means $(1, k)$ $-$localized
$(\epsilon, 1)$ $-$delocalized implies $||v||_\infty^2 \leq \epsilon$
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Showed: If A has girth g then every eigvec is $(d^{g/4}, 1)$-delocalized
Q. What about $\epsilon < 1$?

[Brooks-Lindenstrauss'09]

Thm. Suppose G is $d+1$-regular with girth g and $Av = \lambda v$.

1. If $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ then v is $(\epsilon, \epsilon^2 d^{c\epsilon^2 g})$-delocalized for $\epsilon \in (0,1)$.
2. If $\lambda \notin \left(-2\sqrt{d} - \delta, 2\sqrt{d} - \delta\right)$ then $||v||_\infty^2 \leq d^{-\Omega_\delta(g)}$.

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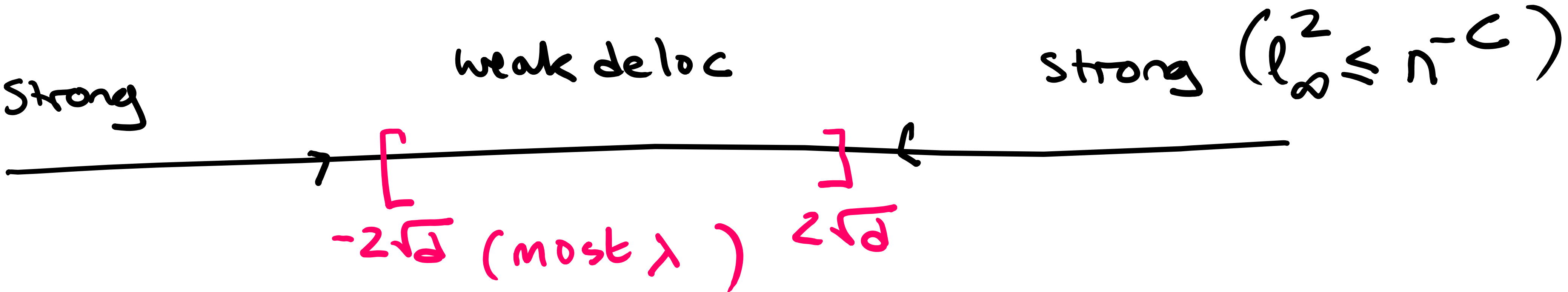
When $g = \Omega(\log_d n)$, (1) implies $(\epsilon, \epsilon^2 n^{\Omega(\epsilon^2)})$-deloc and

$$||v||_\infty^2 \leq (\log_d n)^{-1/2}$$

whereas (2) implies $||v||_\infty^2 \leq n^{-c}$.



Questions

Q1. How does localization depend on the eigenvalue λ?

Q2. How does localization depend on the mass ϵ?

(in [BL'06], exponent of ϵ depends on $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ and Diophantine properties of λ)

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Q3. How do high girth graphs compare to random regular graphs?

(cf. [BHY'16] shows bulk eigenvectors have $||v||_\infty^2 \leq \frac{\log^C(n)}{n}$)

Theorem A [Ganguly-S'18]

Thm. Suppose G is $d+1$-regular with girth g and $Av = \lambda v$.

then v is $(\epsilon, \epsilon d^{\frac{\epsilon g}{4} - 3})$-delocalized for $\epsilon \in (0,1)$.

Improved constant and exponent of ϵ compared to [BL'06] part (1).

Implies $||v||_\infty^2 \leq (\log_d n)^{-1}$.

Contrapositive: (k, ϵ)-localized implies $g \leq 4\log(\frac{k}{\epsilon})/\epsilon + O(1)$.

Proof is a technical improvement of [BL'06] (approx. theory + nonbacktracking walks)

Theorem B [Alon-Ganguly-S'19]

Fix d prime. There is an infinite sequence of $d+1$-regular graphs G_m on m vertices such that:

1. $girth(G_m) \geq \left(\frac{1}{3}\right) \log_d(m)$
2. There is an eigenvector $A_m v = \lambda v$ which is (k, ϵ) −localized for

$$k = O(d^{4\epsilon girth(G_m)}) \quad \forall \epsilon \in (0,1]$$

Implies exponent of ϵ in Theorem A cannot be improved.

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The set of λ attained by the above sequence is dense in $(-2\sqrt{d}, 2\sqrt{d})$.

Implies arithmetic properties of λ do not play a role.

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3. $|\lambda_i(A_m)| \leq 2.12\sqrt{d}$ for all nontrivial adjacency eigenvalues.

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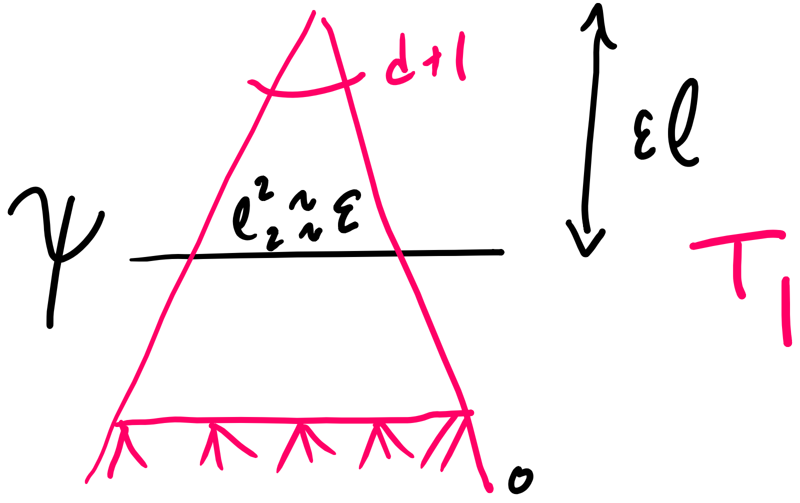
Proof of Theorem B (simplified)

Step 1. Finite $d + 1$-ary tree of depth ℓ
with n leaves.



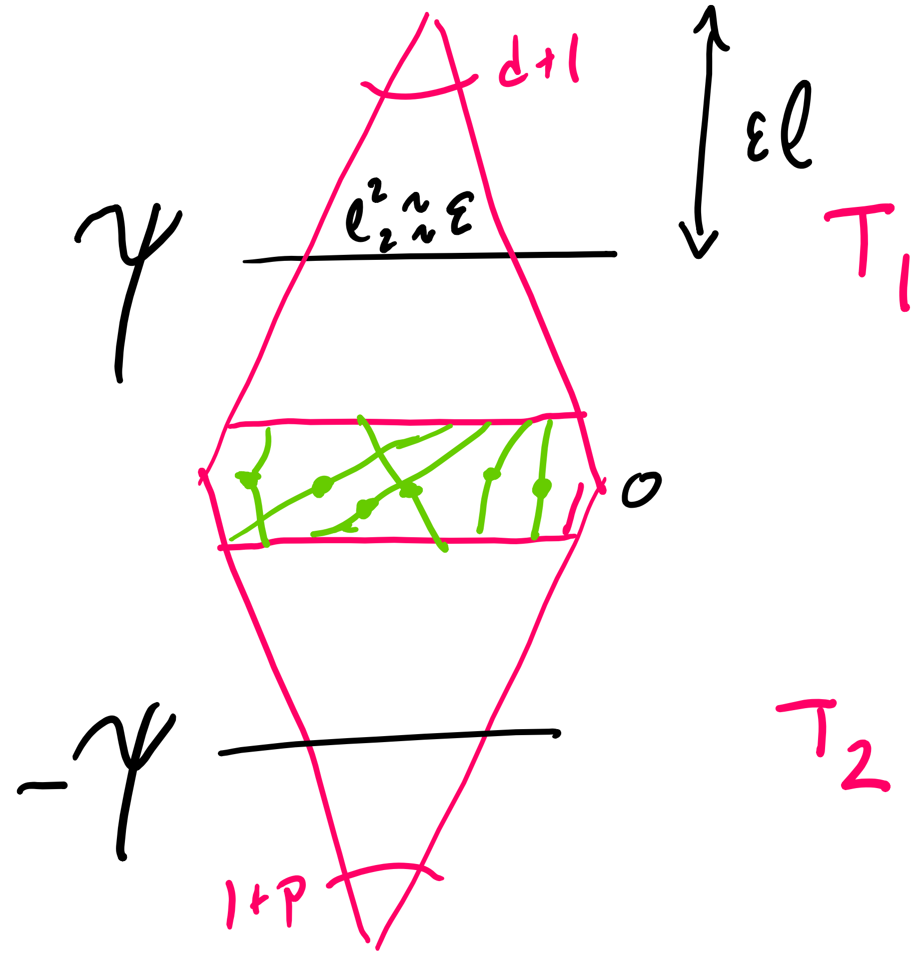
Fact: has many $(\epsilon, d^{\epsilon\ell})$-localized eigenvectors.
equal to zero on the leaves.

Proof of Theorem B

Step 2. Two $d + 1$-ary trees of depth ℓ,

with leaves identifed to maximize

girth [Erdos-Sachs] or [McKay]



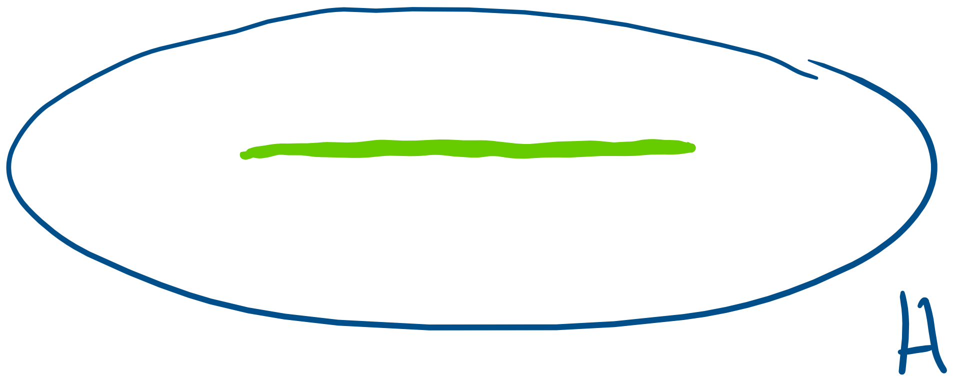
Yields girth $\geq \Omega(\ell) = \Omega(\log_d n)$

Eigenvector equation is satisfied by

Reflecting ψ on the paired tree.

Proof of Theorem B

Step 3. Let H be a $d+1$-regular Ramanujan [LPS,Margulis] graph with n **defects** of degree d at mutual distance $\Omega(\log_d n)$



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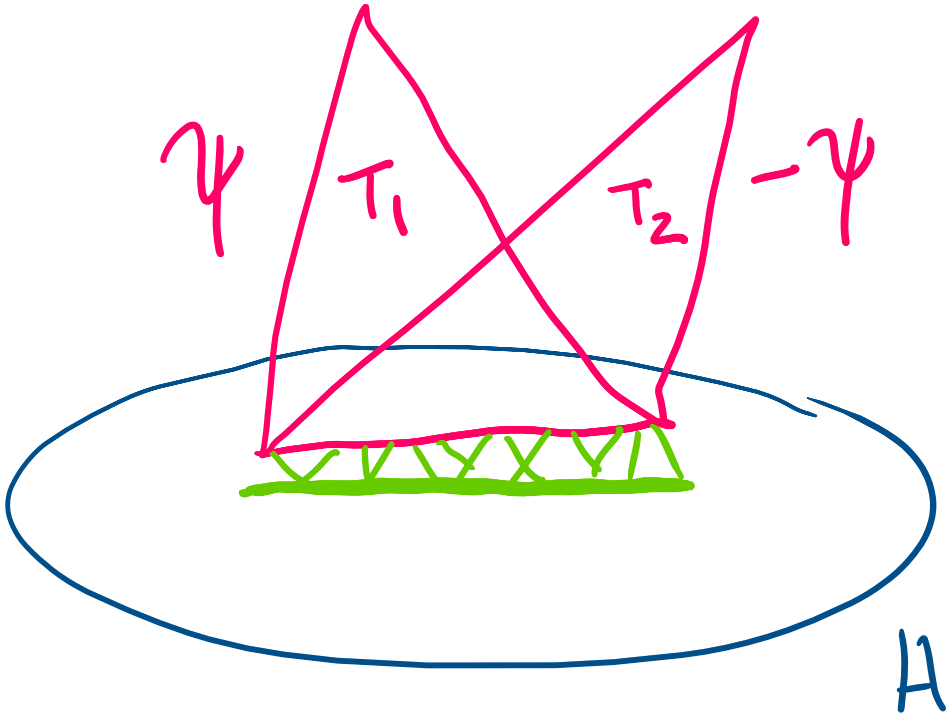

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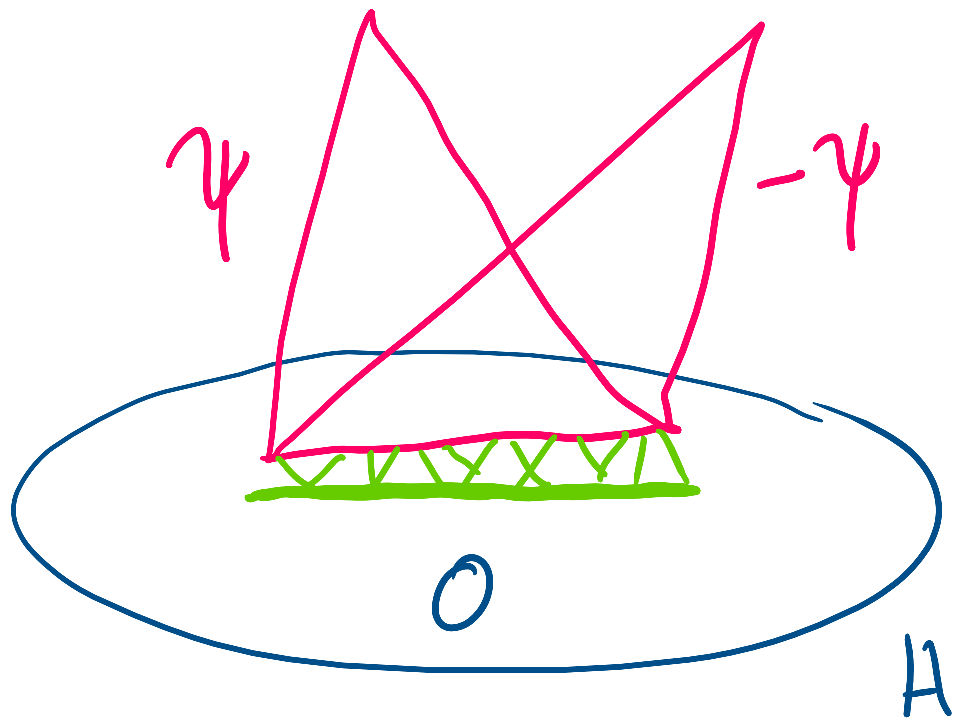
Set eigenvector to zero on H.



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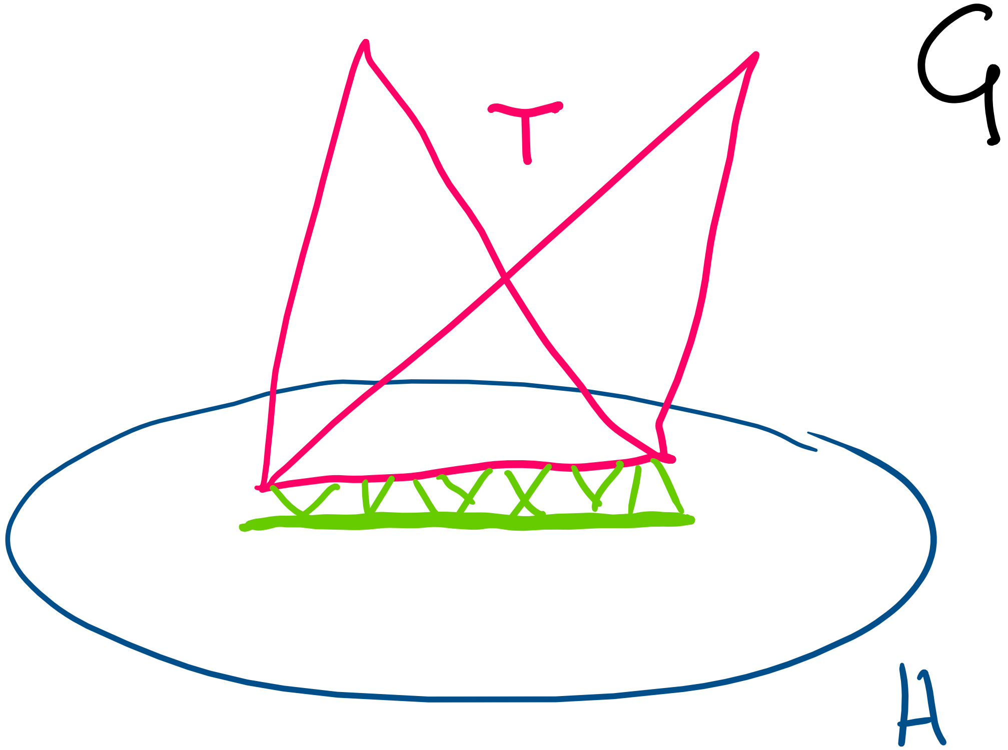
Spectral Gap

Observation: If any eigenvector of G is equal to zero on the **interface** then

$$v^T A_G v = v^T A_T v + v^T A_H v$$
$$\leq 2\sqrt{d}||v_T||^2 + 2\sqrt{d}||v_H||^2 + o(1)$$



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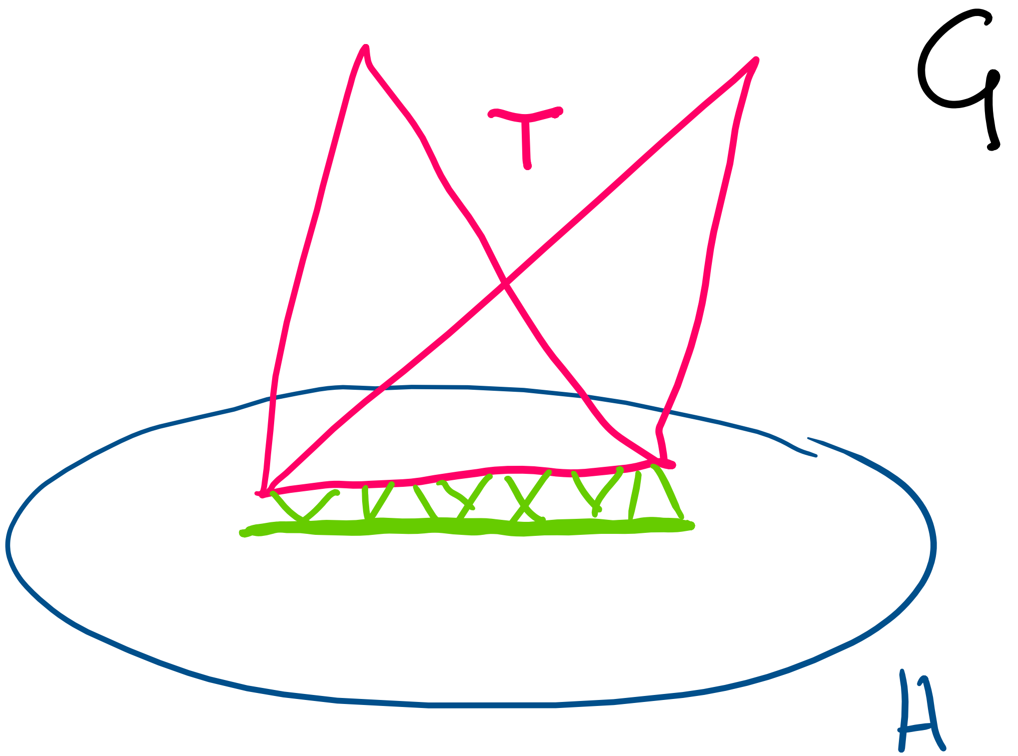

Key Lemma: Any putative non-Ramanujan eigenvector of G must have at most 5% of its mass on the interface.

(high girth + [Kahale'95] argument)

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≤2.04 for d=2. but not Ramanujan due to bad vertex expansion [Kahale]

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The Quantum Ergodicity Angle

[Shnirelman’74] If the geodesic flow on a compact manifold is *ergodic*, then there is a dense subsequence of Laplacian eigenfunctions which is *equidistributed* (a strong notion of delocalization).

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QUE Conjecture [Rudnick-Sarnak]:If the manifold is *negatively curved*, this is true for **all** eigenfunctions. Special case proved by Lindenstrauss.

[Smilansky’07] Study graphs as a simplified model for manifolds.

Quantum Ergodicity on Graphs

[Anantharaman-Le Masson'13] If $G = (V, E)$ is a bounded degree regular *high girth expander* with unit eigenvectors $v_1, \dots, v_n$ then:

$$\max_{|f| \le 1} \sum_i \left| \sum_{x \in V} v_i^2(x) f(x) - \sum_x f(x) \right|^2 = o(n)$$

In fact, they proved this for eigenvectors in any $1/\log(n)$ interval.

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Strongest version of QUE: this is true for *every* eigenvector.

Theorem B disproves this strongest version.

Questions

- Actual Ramanujan graphs in Theorem B?

  Could Ramanujan + High Girth -> Strong delocalization?
- Minimal assumptions for Quantum Ergodicity on Graphs?

  Construct graphs with many localized λ in a small interval?
- More surgery on graphs preserving spectrum [Alon’20], [Paredes’20]
- Use interior eigenvectors to study girth, expansion?

  [cf. Naor’12 for Abelian Cayley Graphs]