# Math 54 Second Midterm Exam, Prof. Srivastava
## October 31, 2018, 5:10pm–6:30pm, 150 Wheeler.

Name: Nikhil Srivastava

SID: ______________________

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If $A$ and $B$ are $n \times n$ square matrices then $(A+B)^2 = A^2 + 2AB + B^2$.

   **Solution:** False. $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$, which is different from $A^2 + 2AB + B^2$ unless $AB = BA$, which does not happen in general.

   (b) If all of the eigenvalues of a matrix are complex (i.e., not real), then it must be invertible.

   **Solution:** True. If all the eigs are complex then in particular none of them are equal to zero (since zero is real), so $Ax = 0$ does not have a nontrivial solution, and $A$ must be invertible.

   (c) If $A$ is an $n \times n$ orthogonal matrix then the RREF of $A$ must have $n$ pivots.

   **Solution:** True. An orthogonal matrix satisfies $A^T A = I$, so $A$ has an inverse, which means that its RREF is equal to the identity, which has $n$ pivots.

   (d) If $A$ is similar to $B$ and $B$ is orthogonal then $A$ must be orthogonal.

   **Solution:** False. Consider $B = (1/\sqrt{2}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, which corresponds to rotation counterclockwise by $\pi/4$ radians, and let

$$A = PBP^{-1} = (1/\sqrt{2}) \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix},$$

   where $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Since the columns of $A$ are not orthonormal, it is not orthogonal.

   *This was one of the hardest questions, and you might wonder how you could come up with such an example. You could first try the algebra: if $A = PBP^{-1}$ and $B^T B = I$ then $A^T A = (PBP^{-1})PBP^{-1} = P^{-T} B^T P^T PBP^{-1}$. This would be equal to the identity if $P^T P = I$ and $P^{-T} P^{-1} = I$, i.e., if $P$ was itself orthogonal. This gives a clue as to what a counterexample could look like; choosing $P$ to be clearly not orthogonal as I did above works.*

   *More conceptually, you could recall that an orthogonal matrix preserves dot products and norms, and corresponds to a high-dimensional 'rotation'. So you are asking if changing basis, rotating, and changing basis back is also a rotation. But this seems geometrically unlikely, since changing basis can distort angles arbitrarily.*

(e) For every subspace $H$ of $\mathbb{R}^n$, there is a matrix $A$ such that $H = \text{Null}(A)$.

> **Solution:** True. Let $v_1, \ldots, v_k$ be a basis for $H^\perp$. Then $H = \{x \in \mathbb{R}^n : x \cdot v_i = 0\text{for } i = 1, \ldots, k\} = \text{Nul}(A)$.

(f) If $\lambda$ is an eigenvalue of $A$ then $\lambda^2$ must be an eigenvalue of $A^2$.

> **Solution:** True. If $Ax = \lambda x$ then $A^2x = A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x$.

(g) If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$ and both are $n \times n$, then $\lambda\mu$ must be an eigenvalue of $AB$.

> **Solution:** False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. The eigenvalues of the product $AB$ are 1 and 8, but $6 = 2 \cdot 3$ is not an eigenvalue.
>
> *This was another hard one. One way to come up with such an answer is to observe that an $n \times n$ matrix has $n$ eigenvalues, but there are $n^2$ possible pairs of eigenvalues of two matrices $A$ and $B$, so if all these pairs have different products all the products cannot possibly be eigenvalues of $AB$.*
>
> *An alternate line of reasoning is to just follow the definitions: If $Ax = \lambda x$ and $Bv = \mu v$ then $ABv = A(\mu v)$, but $v$ is not an eigenvector of $A$, so this vector has no reason to be parallel to $v$.*

(h) The normal equations $A^TA\hat{x} = A^Tb$ always have a unique solution $\hat{x}$.

> **Solution:** False. The normal equations are always consistent (since a least squares problem always has a solution), but if $A^TA$ is not invertible they have infinitely many solutions since every point in $\text{Col}(A)$ is the image of infinitely many $\hat{x}$.

(i) The change of coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ between the bases $\mathcal{B} = \{2e_1, 3e_2\}$ and $\mathcal{C} = \{-3e_1, 4e_2\}$ of $\mathbb{R}^2$ is a diagonal matrix.

> **Solution:** True. Since $[-3e_1]_B = \begin{bmatrix} -3/2 \\ 0 \end{bmatrix}$ and $[4e_2]_B = \begin{bmatrix} 0 \\ 4/3 \end{bmatrix}$.

(j) The set of diagonalizable $2 \times 2$ matrices is a subspace of the set of all $2 \times 2$ real matrices, with scalar multiplication and vector addition defined entrywise.

> **Solution:** False. In order for this to be true, the sum of any two $2 \times 2$ diagonalizable matrices would have to be diagonalizable. But this is false, for instance

for the matrices $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (5 points) A set of three nonzero orthogonal vectors in $\mathbb{R}^2$.

   **Solution:** Does not exist. This is because any set of nonzero orthogonal vectors is linearly independent (by a theorem from class or from the book), and it is not possible to have three linearly independent vectors in $\mathbb{R}^2$ (by another theorem).

   (b) (5 points) A real $3 \times 3$ matrix of rank 2 with only one distinct eigenvalue.

   **Solution:** Consider
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$
   Then $A$ has only one eigenvalue (zero), and two linearly independent columns (the last two, which are already in REF), so it has the desired properties.

   *This was a hard problem, and you might wonder how you could come up with such an example. The first clue is that the matrix has rank less than 3 so it must be singular, which means 0 must be an eigenvalue. Since there is only one distinct eigenvalue, all the eigs must be zero. This is true for any upper triangular matrix with zeros on the diagonal. Filling in the rest of the entries to make the rank large gives many examples.*

   (c) (5 points) A real $3 \times 3$ matrix $A$ along with an isomorphism (i.e. $1-1$ and onto linear transformation) $T : \mathrm{Col}(A) \to \mathrm{Nul}(A)$.

   **Solution:** Does not exist. This is because an isomorphism preserves dimension: since it is $1-1$ and onto, the image of any basis of $\mathrm{Col}(A)$ will be a basis of $\mathrm{Nul}(A)$. Thus we must have $\dim(\mathrm{Col}(A)) = \dim(\mathrm{Nul}(A))$. Now by the rank nullity theorem
$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = 3 = 2\dim(\mathrm{Col}(A)),$$
   which is impossible since 3 is not even.

   (d) (5 points) A real $3 \times 3$ orthogonal matrix $U$ such that $\det(U) = 2$.

**Solution:** Does not exist. If $U$ is orthogonal then $U^TU = I$, so by properties of determinants:

$$\det(U^TU) = \det(U^T)\det(U) = \det(U)^2 = 1.$$

Thus $\det(U) \neq 2$.
An alternate proof is the geometric fact that $|\det(U)|$ is the volume of the parallelopiped spanned by the columns of $U$, which is equal to one if the columns are orthonormal (since the parallelopiped is a high dimensional cube).

3. Let $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be the vector space of $2 \times 2$ real matrices with entrywise scalar mutliplication and addition. Let

$$H = \{X \in M_2 : Xv = 0\}, \text{where } v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) (4 points) Show that $H$ is a subspace of $M_2$.

**Solution:** $H$ is already a subset of $M_2$, so to verify that it's a subspace we check three properties:

- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v = 0$, so $0 \in H$.
- If $X \in H$ then for any scalar $c$, $(cX)v = c(Xv) = c(0) = 0$, so $cX \in H$.
- If $X, Y \in H$ then $(X + Y)v = Xv + Yv = 0 + 0 = 0$, so $X + Y \in H$.

(b) (7 points) Find a basis for $H$.

**Solution:** Observe that $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in H$ if and only if

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_3 - x_4 \end{bmatrix} = 0,$$

i.e., iff $x_1 = x_2$ and $x_3 = x_4$, which happens precisely when each row of $X$ has only one distinct entry. Thus,

$$H = \{ \begin{bmatrix} x_1 & x_1 \\ x_3 & x_3 \end{bmatrix} : x_1, x_3 \in \mathbb{R}\} = \{x_1 A + x_3 B : x_1, x_3 \in \mathbb{R}\},$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. These vectors are linearly independent since they are not zero and neither one is a multiple of the other, and they span $H$, so they are a basis of $H$.

(c) (1 point) What is the dimension of $H$?

**Solution:** Since the basis above has two vector, the dimension is 2.

4. (12 points) Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of real polynomials in $t$ of degree at most 2, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(p) = 2t \cdot \frac{d}{dt} p(t) + p(-1),$$

where $p(1)$ means the evaluation of $p$ at $t = -1$. Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of $T$ relative to $\mathcal{B}$ is diagonal? If so, find such a basis. If not, explain why.

**Solution:** This is very similar to question 3 on practice midterm 2 #1. First we find the matrix of $T$ with respect to the standard basis $E = \{1, t, t^2\}$ of $\mathbb{P}_2$ (we could work with any basis but the standard basis is usually simplest). We compute

$$[T(1)]_E = [2t \cdot \frac{d(1)}{dt} + 1]_E = [1]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[T(t)]_E = [2t \cdot \frac{d(t)}{dt} + (-1)]_E = [-1 + 2t]_E = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix},$$

$$[T(t^2)]_E = [2t \cdot \frac{d(t^2)}{dt} + (-1)^2]_E = [1 + 4t^2]_E = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

The matrix $[T]_E$ is the matrix with these coordinate vectors as columns, so we have

$$[T]_E = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

This matrix is upper triangular so its eigenvalues are $1, 2, 4$. Since these are distinct, we know $[T]_E$ will be diagonalizable. Its eigenspaces are:

$$E_1 = \text{Nul}([T]_E - I) = \text{span}\{e_1\},$$

$$E_2 = \text{Nul}([T]_E - 2I) = \text{span}\{e_1 - e_2\},$$

$$E_4 = \mathrm{Nul}([T]_E - 4I) = \mathrm{span}\{e_1 + 3e_3\},$$

where $e_1, e_2, e_3$ are the standard basis of $\mathbb{R}^3$. Thus we have $[T]_E e_1 = e_1, [T]_E(e_1 - e_2) = 2(e_1 - e_2)$, and $[T]_E(e_1 + 3e_3) = 4(e_1 + 3e_3)$. By the correspondence principle, this means that $T(1) = 1, T(1-t) = 2(1-t)$, and $T(1+3t^2) = 4(1+3t^2)$. Thus, the matrix of $T$ with respect to the basis $\mathcal{B} = \{1, 1-t, 1+3t^2\}$, is diagonal and equal to

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

5. (7 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 11 & 12 \\ 0 & 0 & 0 & 3 & 14 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Show $A$ is invertible. Find $\det(A^{-1})$.

**Solution:** Applying the row operation $R_2 \leftarrow R_1 + R_2$, we see that $A$ is row equivalent to:

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 8 & 10 & 12 & 14 \\ 0 & 0 & 1 & 11 & 12 \\ 0 & 0 & 0 & 3 & 14 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

This matrix is in REF and has a pivot in every column, so $A$ must be invertible. Since the row op we applied does not change the determinant, we have

$$\det(A) = \det(B) = 1 \cdot 8 \cdot 1 \cdot 3 \cdot 5 = 120,$$

because $B$ is upper triangular. We now have

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{120}.$$

6. (7 points) Find a linear combination of $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ which is orthogonal

to $w = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, or explain why no such linear combination exists.

**Solution:** A linear combination of $v_1$ and $v_2$ is a vector of type $v = c_1v_1 + c_2v_2$. We want to find $c_1$ and $c_2$ such that

$$v \cdot w = (c_1v_1 + c_2v_2) \cdot w = c_1(v_1 \cdot w) + c_2(v_2 \cdot w) = c_1(2+0-1) + c_2(2-2+3) = c_1 + 3c_2 = 0.$$

This is a linear system with 2 variables and one equation, which has infinitely many nonzero solutions, given by

$$\mathrm{Nul}(\begin{bmatrix} 1 & 3 \end{bmatrix}) = \mathrm{span}\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}.$$

In particular, the vector $3v_1 - v_2$ is a nonzero linear combination of $v_1$ and $v_2$ orthogonal to $w$ (it is not zero since $v_1$ and $v_2$ are linearly independent).

7. Let

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

(a) (8 points) Find a least squares solution to $Ax = b$, i.e., a vector $\hat{x} \in \mathbb{R}^2$ minimizing $\|A\hat{x} - b\|$.

Computing $A^TA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$, the normal eqn is $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\hat{x} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$

Row reducing gives the unique soln $\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

(b) (3 points) Using your answer to (a), or otherwise, find the orthogonal projection $\hat{b}$ of $b$ onto the column space of $A$, i.e., $\hat{b} = \mathrm{Proj}_{\mathrm{Col}(A)}(b)$.

The projection is $\hat{b} = A\hat{x}$

$= \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

(c) (3 points) Find the projection of $b$ onto $\text{Col}(A)^\perp$. What is the distance of $b$ from $\text{Col}(A)$?

The projection of $\underline{b}$ onto $\text{Col}(A)^\perp$ is

$\underline{b} - \hat{\underline{b}} = \underline{0}$ // Thus the

distance of $\underline{b}$ from $\text{Col}(A)$ is $\|\underline{b} - \hat{\underline{b}}\| = 0$.

8. (8 points) Find an orthogonal basis for the row space of the matrix

$$A = \begin{bmatrix} 2 & -5 & 1 \\ 4 & -10 & 2 \\ 4 & -1 & 2 \\ -4 & -8 & -2 \end{bmatrix}.$$

Thus, an orthogonal basis is $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$

The row space is spanned by $\underline{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 4 \\ -10 \\ 2 \end{bmatrix}$

$\underline{x}_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ $\underline{x}_4 = \begin{bmatrix} -4 \\ -8 \\ -2 \end{bmatrix}$.

Running Gram Schmidt, we get an orthogonal basis:

$\underline{w}_1 = \underline{x}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$.

$\underline{w}_2 = \underline{x}_2 - \frac{\underline{w}_1 \cdot \underline{x}_2}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1 = \begin{bmatrix} 4 \\ -10 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \underline{0} \Rightarrow$ ignore

$\underline{w}_3 = \underline{x}_3 - \frac{\underline{w}_1 \cdot \underline{x}_3}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$



$\underline{w}_4 = \underline{x}_4 - \frac{\underline{w}_1 \cdot \underline{x}_4}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1 - \frac{\underline{w}_3 \cdot \underline{x}_4}{\underline{w}_3 \cdot \underline{w}_3} \underline{w}_3 \overset{\text{(skipped arithmetic)}}{=} \underline{0}$ // $\Rightarrow$ ignore