Math 54 Second Midterm Exam, Prof. Srivastava
October 31, 2018, 5:10pm–6:30pm, 150 Wheeler.

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Sign here: ______________________________

Name of the student to your left: ______________________________

Name of the student to your right: ______________________________

Question	Points
1	20
2	20
3	12
4	12
5	7
6	7
7	14
8	8
Total:	100

Do not turn over this page until your instructor tells you to do so.

Name and SID: ______________________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If A and B are $n \times n$ square matrices then $(A+B)^2 = A^2 + 2AB + B^2$. **T** **F**

   (b) If all of the eigenvalues of a matrix are complex (i.e., not real), then it must be invertible. **T** **F**

   (c) If A is an $n \times n$ orthogonal matrix then the RREF of A must have n pivots. **T** **F**

   (d) If A is similar to B and B is orthogonal then A must be orthogonal. **T** **F**

   (e) For every subspace H of $\mathbb{R}^n$, there is a matrix A such that $H = \text{Null}(A)$. **T** **F**

   (f) If λ is an eigenvalue of A then λ^2 must be an eigenvalue of A^2. **T** **F**

   (g) If λ is an eigenvalue of A and μ is an eigenvalue of B and both are $n \times n$, then $\lambda\mu$ must be an eigenvalue of AB. **T** **F**

   (h) The normal equations $A^T A\hat{x} = A^T b$ always have a unique solution $\hat{x}$. **T** **F**

   (i) The change of coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ between the bases $\mathcal{B} = \{2e_1, 3e_2\}$ and $\mathcal{C} = \{-3e_1, 4e_2\}$ of $\mathbb{R}^2$ is a diagonal matrix. **T** **F**

   (j) The set of diagonalizable 2×2 matrices is a subspace of the set of all 2×2 real matrices, with scalar multiplication and vector addition defined entrywise. **T** **F**

2. Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

   (a) (5 points) A set of three nonzero orthogonal vectors in $\mathbb{R}^2$.

   (b) (5 points) A real 3×3 matrix of rank 2 with only one distinct eigenvalue.

(c) (5 points) A real 3×3 matrix A along with an isomorphism (i.e. $1-1$ and onto linear transformation) $T : \mathrm{Col}(A) \to \mathrm{Nul}(A)$.

(d) (5 points) A real 3×3 orthogonal matrix U such that $\det(U) = 2$.

[Scratch Paper 1]

3. Let $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be the vector space of 2×2 real matrices with entrywise scalar mutliplication and addition. Let

$$H = \{X \in M_2 : Xv = 0\}, \text{where } v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) (4 points) Show that H is a subspace of M_2.

(b) (7 points) Find a basis for H.

(c) (1 point) What is the dimension of H?

4. (12 points) Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of real polynomials in t of degree at most 2, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(p) = 2t \cdot \frac{d}{dt} p(t) + p(-1),$$

where $p(1)$ means the evaluation of p at $t = -1$. Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T relative to $\mathcal{B}$ is diagonal? If so, find such a basis. If not, explain why.

5. (7 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 11 & 12 \\ 0 & 0 & 0 & 3 & 14 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Show A is invertible. Find $\det(A^{-1})$.

6. (7 points) Find a linear combination of $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ which is orthogonal to $w = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, or explain why no such linear combination exists.

7. Let

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

(a) (8 points) Find a least squares solution to $Ax = b$, i.e., a vector $\hat{x} \in \mathbb{R}^2$ minimizing $\|A\hat{x} - b\|$.

(b) (3 points) Using your answer to (a), or otherwise, find the orthogonal projection $\hat{b}$ of b onto the column space of A, i.e., $\hat{b} = \mathrm{Proj}_{\mathrm{Col}(A)}(b)$.

(c) (3 points) Find the projection of b onto $\mathrm{Col}(A)^{\perp}$. What is the distance of b from $\mathrm{Col}(A)$?

8. (8 points) Find an orthogonal basis for the row space of the matrix

$$A = \begin{bmatrix} 2 & -5 & 1 \\ 4 & -10 & 2 \\ 4 & -1 & 2 \\ -4 & -8 & -2 \end{bmatrix}.$$