Math 54 Fall 2016 Practice Midterm 2 Solutions

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50 minutes, closed book, closed notes

1. True or False (no need for justification):

   (a) If V is a vector space with a finite basis then V is isomorphic to $\mathbb{R}^n$ for some n.
   True. If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis then the coordinate mapping $x \mapsto [x]_\mathcal{B}$ is an isomorphism from V to $\mathbb{R}^n$.

   (b) The system $A^T Ax = A^T b$ is consistent for all A and b.
   True. This is because the least squares solution to $Ax = b$ (i.e., the x minimizing $\|Ax - b\|$) always satisfies the normal equations.

   (c) If A and B are similar and A is diagonalizable then B must be diagonalizable.
   True. If A is diagonalizable then $A = PDP^{-1}$ for some diagonal P. Since A and B are similar, we have $B = QAQ^{-1}$ for some Q. Combining these, we get $B = QPDP^{-1}Q^{-1} = (QP)D(QP)^{-1}$, so B is diagonalizable as well.

   (d) The rank of a square matrix is equal to the number of nonzero eigenvalues (counted with multiplicity).
   False. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has only zero eigenvalues, but rank 1.
   This was one of the the hardest questions on the exam, because it is almost true. In particular, if A is $n \times n$ and diagonalizable, then the dimension of the nullspace of A is equal to the multiplicity m of zero as an eigenvalue (since the dimension of E_0 is equal to this multiplicity). Thus, the number of nonzero eigenvalues of A is $n - m$, which is also equal to the rank since $m + \operatorname{rank}(A) = n$.

   (e) If x and y are arbitrary nonzero vectors in $\mathbb{R}^n$ then there is a basis B of $\mathbb{R}^n$ such that $[x]_B = y$.
   True. This was the hardest one. The question is asking whether there is an invertible matrix P_B such that $P_B y = x$ (the columns of such a matrix would then form the basis B). To find such a matrix, first choose any basis C whose first vector is x, namely $c_1 = x, c_2, c_3, \ldots, c_n$, and let P_C be the matrix whose columns are $c_1, \ldots, c_n$. Similarly, let D be a basis whose first vector is $d_1 = y, d_2, \ldots, d_n$. (Note that for any vector, there is a basis containing that vector). We now have $[x]_C = P_C^{-1}x = e_1$ and $[y]_D = P_D^{-1}y = e_1$, so

$$P_D^{-1}y = P_C^{-1}x.$$

Multiplying by P_D gives $y = P_D P_C^{-1} x$, so the desired P_B is just $P_D P_C^{-1}$.

(f) Every eigenvalue of a square matrix A is a pivot of A in the reduced row echelon form of A.

False. The RREF of every invertible matrix is the identity, but not every invertible matrix has all eigenvalues equal to one.

(g) If A is a square matrix then A and A^T have the same eigenvalues.

True. This is because $\det(A - tI) = \det((A - tI)^T) = \det(A^T - tI)$, which may be seen by using the cofactor expansion of the determinant. So A and A^T have the same characteristic polynomial, which means they have the same eigenvalues.

(h) An upper triangular matrix is always diagonalizable.

False. Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(i) An set of orthogonal vectors is always linearly independent.

False. An orthogonal set can contain a zero vector, which would make it automatically linearly dependent.

I did not intend to ask such a trick question, I missed the word 'nonzero', which is a typo. I really should have asked: is a set of nonzero orthogonal vectors linearly independent? In this case the answer is yes.

(j) If $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent and W is a subspace of $\mathbb{R}^n$ then $\mathrm{Proj}_W(v_1), \dots, \mathrm{Proj}_W(v_k)$ must also be linearly independent.

False. Consider the subspace $W = \mathrm{span}\{e_1\}$ of $\mathbb{R}^2$. Then $\{e_1, e_2\}$ is linearly independent, but $\mathrm{Proj}_W(e_2) = 0$, so the set of projections is not.

2. Let $V = \mathbb{R}^{3\times 3}$ denote the vector space of real 3×3 matrices with addition and scalar multiplication defined entrywise. Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

and consider the subset

$$W = \{X \in V : XM = MX\}$$

of matrices in V which commute with M. Is W a subspace of V? If so, prove it. If not, explain why.

Solution: This question just involves knowing how to check whether a given set is a subspace. Recall that this requires satisfying three properties: (a) The zero matrix $0 \in W$, since $0M = 0 = M0$. (b) Assume $X, Y \in W$, which means that $XM = MX$ and $YM = MY$. Adding these equations, we find that

$$XM + YM = MX + MY.$$

By the distributive property from matrix algebra, this is equivalent to

$$(X+Y)M = M(X+Y),$$

which implies that $X+Y \in W$. (c) Assume $X \in W$ and $c \in \mathbb{R}$. We know $XM = MX$. Multiplying by C gives $cXM = cMX$. Again by matrix algebra, this is just $(cX)M = M(cX)$, so $cX \in W$.

Since W is a subset of V, contains zero, and is closed under addition and scalar multiplication, W is a subspace of V.

Note that the actual details of the matrix M did not matter in this case.

3. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2\}$ be the vector space of polynomials of degree at most 2 with coefficient-wise operations, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by
$$T(q) = \frac{d^2}{dt^2}q + t \cdot \frac{d}{dt}q + 3q.$$
Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T with respect to $\mathcal{B}$ is diagonal? If so, find such a basis as well as the corresponding matrix. If not, explain why.

Solution. This question is asking whether T is diagonalizable, To figure this out, we first find the matrix of T with respect to any basis; let's use the standard basis $E = \{1, t, t^2\}$ of $\mathbb{P}_2$. Computing

$$[T(1)]_E = [0+0+3]_E = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},$$

$$[T(t)]_E = [0+t+3t]_E = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix},$$

$$[T(t^2)]_E = [2+2t^2+3t^2]_E = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix},$$

we find that the matrix is given by

$$A = [T]_E = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Since this matrix is upper triangular, its eigenvalues are just the diagonal entries, which are $3, 4, 5$. Since these are distinct, we know that the matrix must be diagonalizable. To find the basis, we find the corresponding eigenvectors:

$$E_3 = Null(A - 3I) = \text{span}\{e_1\},$$

$$E_4 = Null(A - 4I) = \text{span}\{e_2\},$$

$$E_5 = Null(A - 5I) = Null \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}.$$

Thus, $Ae_1 = 3e_1, Ae_2 = 4e_2$, and $A(e_1 + e_3) = 5(e_1 + e_3)$. Applying the correspondence principle, we find that, $T(1) = 3, T(t) = 4t, T(1 + t^2) = 5(1 + t^2)$. Thus the matrix of T with respect to the basis $B = \{1, t, 1 + t^2\}$ is diagonal, and equal to

$$[T]_B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Alternatively, we may diagonalize $[T]_E = A = PDP^{-1} = P_B D P_B^{-1}$ *for* D *as above and* P_B *containing* $e_1, e_2, (e_1 + e_3)$ *as columns. Multiplying on the left by* P_B^{-1} *and on the right by* P_B *we have* $P_B^{-1}[T]_E P_B = D$*. Recalling that* $P_B^{-1} = P_{B\leftarrow E}$ *and* $P_B = P_{E\leftarrow B}$*, we have*

$$D = P_{B\leftarrow E}[T]_E P_{E\leftarrow B} = [T]_B,$$

as desired.

4. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Find the eigenvalues of A. Compute A^{11}.

Solution: We begin by computing the characteristic polynomial:

$$\chi_A(t) = \det \begin{bmatrix} 1 - t & 1 \\ -2 & 4 - t \end{bmatrix} = (1 - t)(4 - t) + 2 = t^2 - 5t + 6 = (t - 2)(t - 3),$$

so the eigenvalues are $2, 3$. The corresponding eigenspaces are

$$E_2 = Null \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} = \text{span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \},$$

$$E_3 = Null \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} = \text{span}\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \},$$

so we may diagonalize A as $A = PDP^{-1}$ with $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

We now have

$$A^{11} = PD^{11}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^{11} & 0 \\ 0 & 3^{11} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

I could simplify this, but I won't. I'll give you easier numbers on the exam if such a question arises, and in any case, it will be fine to leave expressions such as $2 \cdot 2^{11} - 3^{11}$ *unsimplified.*

5. Consider the vectors

$$x_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 12 \\ -4 \\ 7 \\ 1 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix},$$

and let $W = \text{Span}\{x_1, x_2\}$. Find vectors w and z such that $w \in W, z \in W^\perp$, and $y = w + z$. What is the distance between y and the closest point in W to y?

Solution: We first check whether the given vectors are orthogonal: $x_1 \cdot x_2 = 24 + 0 - 7 - 3 = 14 \neq 0$, so they aren't. So we use Gram-Schmidt to find an orthogonal basis for W:

$$u_1 = x_1$$

$$u_2 = x_2 - \frac{x_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 12 \\ -4 \\ 7 \\ 1 \end{bmatrix} - \frac{14}{14} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \\ 8 \\ 4 \end{bmatrix}.$$

We now use the decomposition theorem to find the desired vectors:

$$w = \text{Proj}_W(y) = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{4+0+0+3}{14} u_1 + \frac{20-16+0-4}{100+16+64+16} u_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix},$$

$$z = y - w = \begin{bmatrix} 1 \\ 4 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Since w is the closest point in W to y, the distance is given by

$$\|z\| = \sqrt{1 + 16 + 1/4 + 1/4} = \sqrt{35/2}.$$

6. Let W be a subspace of $\mathbb{R}^n$ and let P be the standard matrix of the projection onto W, i.e., $P = [\text{Proj}_W]$ where $\text{Proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation which projects onto W. (a) Show that $P^2 = P$. (b) Use this to show that the eigenvalues of P are all either 0 or 1. (c) What is the eigenspace corresponding to the eigenvalue 1 of P?

Solution: (a) There are several ways to show this:

Conceptual Way: Observe that for every vector $w \in W$, we have $w = w + 0$ where $w \in W$ and $0 \in W^\perp$. Since the decomposition theorem says this is unique, this implies that $\text{Proj}_W(w) = w$ for every $w \in W$. Thus, for every vector $x \in \mathbb{R}^n$ we have $\text{Proj}_W(\text{Proj}_W(x)) = \text{Proj}_W(x)$ since $\text{Proj}_W(x) \in W$. Thus, the linear transformations

$\mathrm{Proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathrm{Proj}_W \circ \mathrm{Proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ are the same, which means that their standard matrices must be the same. But since $[\mathrm{Proj}_W] = P$ and composition corresponds to matrix multiplication, this means that $P^2 = P$.

Algebraic way: Let $u_1, \ldots, u_k$ be an orthonormal basis for W (by Gram-Schmidt such a basis exists) and let U be the $n \times k$ matrix with $u_1, \ldots, u_k$ as columns. Recall from lecture that $P = UU^T$ and $U^T U = I_k$. Thus we have $P^2 = (UU^T)(UU^T) = U(U^T U)U^T = U(I)U^T = UU^T = P$.

(b) Assume λ is an eigenvalue of P. Then $Px = \lambda x$ for some nonzero vector x. Since $P^2 = P$ we also have

$$Px = P^2 x = P(Px) = P(\lambda x) = \lambda P x = \lambda^2 x.$$

Thus we must have $\lambda^2 x = \lambda x$, which since $x \neq 0$ means $\lambda^2 = \lambda$. Thus we must have $\lambda = 0$ or 1.

(c) The eigenspace corresponding to $\lambda = 1$ is the set of vectors x such that $Px = x$, which means $\mathrm{Proj}_W(x) = x$, which happens if and only if $x \in W$. Thus the eigenspace is simply equal to W.