Math 54 Second Midterm Exam, Prof. Srivastava
October 31, 2016, 4:10pm–5:00pm, 155 Dwinelle Hall.

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. This exam includes one page of scratch paper, which must be submitted but will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit.

Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

UC Berkeley Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

Sign here: ______________________________

Name of the student to your left: ______________________________
Name of the student to your right: ______________________________

Question	Points
1	20
2	15
3	10
4	7
5	8
Total:	60

Do not turn over this page until your instructor tells you to do so.

Name and SID: ____________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) If A is a square matrix then $\text{Col}(A) \cap \text{Null}(A) = \{0\}$. **T** **F**

   (b) If $v \in \text{Span}\{w, a\}$ and $w \in \text{Span}\{a, b\}$ then $v \in \text{Span}\{a, b\}$, where v, w, a, b are vectors in a vector space. **T** **F**

   (c) If V and W are $n-$dimensional vector spaces then there is a linear transformation $T : V \to W$ which is both 1-1 and onto. **T** **F**

   (d) If V is an $n-$dimensional vector space then every linearly independent subset of V contains at most n vectors. **T** **F**

   (e) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, $[T]$ is the standard matrix of T, and $[T]_{\mathcal{B}}$ is the matrix of T with respect to another basis $\mathcal{B}$, then $\det([T]) = \det([T]_{\mathcal{B}})$. **T** **F**

   (f) If λ is an eigenvalue of A and μ is an eigenvalue of B then $\lambda + \mu$ is an eigenvalue of $A + B$. **T** **F**

   (g) If A is a 4×4 real matrix with two distinct real eigenvalues and two complex eigenvalues, then it must be diagonalizable over $\mathbb{C}$. **T** **F**

   (h) If W is a subspace of $\mathbb{R}^n$ and x is a vector in $\mathbb{R}^n$ such that $\text{Proj}_W(x) = 0$ then $x \in W^\perp$. **T** **F**

   (i) Let W be a subspace of $\mathbb{R}^n$ and let $P = [\text{Proj}_W]$ be the $n \times n$ matrix of the orthogonal projection onto W. Then every column of P is an element of W. **T** **F**

   (j) If x, y, z are vectors in $\mathbb{R}^3$ such that $x \cdot y = 0$ and $y \cdot z = 0$ then it follows that $x \cdot z = 0$. **T** **F**

2. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ denote the vector space of polynomials of degree at most 2 with coefficient-wise addition and scalar multiplication. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by

$$T(q) = \begin{bmatrix} q(2) \\ q(-3) \end{bmatrix},$$

where $q(2)$ means the polynomial q evaluated at 2.

(a) (5 points) Find the matrix of T with respect to the bases $\mathcal{B} = \{1, t+1, t^2+t\}$ of $\mathbb{P}_2$ and $E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

(b) (5 points) Find a basis for the Kernel (i.e., Null Space) of T. Explain whatever method you use.

(c) (2 points) Is T 1-1? Why or why not?

(d) (3 points) Let $\mathcal{C} = \{1, t, t^2\}$. Find the change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}.$$

(a) (2 points) Show that A is invertible.

(b) (5 points) Find the eigenvalues of A^{-1}. Explain how you got your answer. (hint: there is a faster way than computing the inverse)

(c) (3 points) Is $(A^{-1})^2$ diagonalizable? Explain why or why not.

4. (7 points) Consider the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A. Find a diagonal matrix D an an invertible matrix P such that $AP = PD$, or explain why no such matrices exist.

Name and SID: ______________________________

5. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 6 \\ 1 & 2 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

(a) (5 points) Find a least squares solution to $Ax = b$, i.e., a vector $\hat{x} \in \mathbb{R}^3$ minimizing $\|A\hat{x} - b\|$.

(b) (3 points) Using your answer to (a), or otherwise, find the orthogonal projection $\hat{b}$ of b onto the column space of A, i.e., $\hat{b} = \mathrm{Proj}_{\mathrm{Col}(A)}(b)$.

Name and SID: ____________________

[Scratch Paper]